10.4 Continuous Random Variables and the Normal Distribution

Continuous *normal* distribution of random variable X, defined on interval $(-\infty, \infty)$, has density[11] with parameters μ and σ,

$$f(x) = \frac{1}{\sigma\sqrt{2\pi}} e^{-(1/2)[(x-\mu)/\sigma]^2}$$

and expected value (mean), variance and standard deviation,

$$E(X) = \mu, \quad V(X) = \sigma^2, \quad \sigma = \sqrt{V(X)}.$$

A normal random variable, X, may be transformed to a *standard* normal, Z,

$$f(z) = \frac{1}{\sqrt{2\pi}} e^{-y^2/2},$$

where $\mu = 0$ and $\sigma = 1$ using following equation,

$$Z = \frac{X - \mu}{\sigma}.$$

We discuss the Empirical rule: *if* data is normal,

- 68% of data falls within one SD of average,
- 95% of data falls within *two* SDs of average,
- 99.7% of data falls within *three* SDs of average.

Exercise 10.4(Continuous Random Variables and the Normal Distribution)

1. *Normal: temperature.*
   In Westville, in January, temperature, Z, is assumed *standard* normally distributed with mean $\mu = 0^o$ and standard deviation (SD) $\sigma = 1^o$, whereas in February, temperature, X, is assumed normally distributed with mean $\mu = 5^o$ and standard deviation $\sigma = 0.5^o$

[11]Normal probability density function (pdf) is *not* directly used; the TI-84+ calculator is used instead.

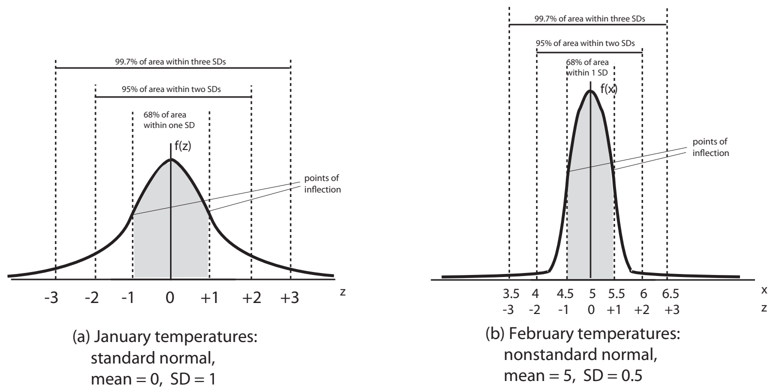

Figure 10.14 (Graphs of normal probability distributions.)

(a) Both (any) normal distributions above are **skewed right** / **symmetric** / **skewed left**.

(b) Shape of both (any) normal are **triangular** / **bell–shaped** / **rectangular**.

(c) Total area (probability) under both (any) normal is **50%** / **75%** / **100%** / **150%**.

(d) *Standard* normal (January) centered at mean (average) temperature $\boldsymbol{\mu = 0^o}$ / $\boldsymbol{\mu = 5^o}$.

(e) Normal (February) centered at mean (average) temperature $\boldsymbol{\mu = 0^o}$ / $\boldsymbol{\mu = 5^o}$.

(f) **True** / **False** Mean = median = mode for both (any) normals.

(g) *Standard* normal (January) has SD in temperature of $\boldsymbol{\sigma = 0.5^o}$ / $\boldsymbol{\sigma = 1^o}$.

(h) Normal (February) has SD in temperature of $\boldsymbol{\sigma = 0.5^o}$ / $\boldsymbol{\sigma = 1^o}$.

(i) *Z-score* converts temperatures to standard ones; for example, February (normal) $x = 6^o$ is "equivalent" to January (standard normal)

$$z = \frac{x - \mu_X}{\sigma_X} = \frac{6 - 5}{0.5} =$$

$\boldsymbol{1^o}$ / $\boldsymbol{2^o}$ / $\boldsymbol{3^o}$ in the sense both $x = 6^o$ and $z = 2^0$ are two SDs above average temperature. The z is a z-score.

(j) *Points of inflection* for both (any) normal occur at (choose *two*!) $\boldsymbol{\mu - \sigma}$ / $\boldsymbol{\mu + \sigma}$ / $\boldsymbol{\mu + 2\sigma}$.

(k) **True** / **False** According to empirical rule, 68%, 95% and 99.7% of probability in both (all) normals is within 1, 2 and 3 SDs, respectively, of mean.

(l) **True** / **False** Since normal has *infinite* bounds, normal function approaches but never equals zero.

2. *Normal again: IQs.*
IQ scores for 16 year olds and 20 year olds described by normal distributions.

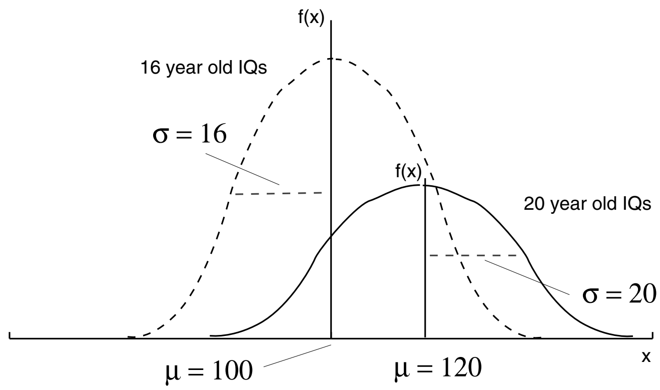

Figure 10.15 (Normal distributions of IQ scores)

(a) Mean IQ score for 20 year olds
$\mu =$ **100** / **120** / **124** / **136**.

(b) Average (or mean) IQ score for 16 year olds
$\mu =$ **100** / **120** / **124** / **136**.

(c) Standard deviation in IQ score for 20 year olds
$\sigma =$ **16** / **20** / **24** / **36**.

(d) Standard deviation in IQ score for 16 year olds
$\sigma =$ **16** / **20** / **24** / **36**.

(e) Normal for 20 year old IQ scores is
broader than / **as wide as** / **narrower than**
normal for 16 year old IQ scores.

(f) Normal for 20 year old IQ scores is
shorter than / **as tall as** / **taller than**
normal for 16 year old IQ scores.

(g) Total area (probability) under normal for 20 year old IQ scores is
smaller than / **same as** / **larger than**
area (probability) under normal for 16 year old IQ scores.

(h) Normal distributions for IQ scores for 20 year old IQ scores and for 16 year old IQ scores are **both** / **neither** standard normals.

(i) A 20 year old IQ score of $x = 140$ is

$$z = \frac{x - \mu_{20}}{\sigma_{20}} = \frac{140 - 120}{20} =$$

1 / **2** / **3** SDs above average 20 year old IQ score.

(j) A 16 year old IQ score of $x = 84$ is

$$z = \frac{x - \mu_{16}}{\sigma_{16}} = \frac{84 - 100}{16} =$$

$\mathbf{-1}$ / $\mathbf{-2}$ / $\mathbf{-3}$ SDs *below* average 16 year old IQ score.

(k) **True** / **False**. z-scores allow comparison of *position* of data points in different data sets, data sets with different averages and SDs.

(l) If $z = \frac{x-\mu}{\sigma}$, then $x = z\sigma + \mu$, so a 16 year old with IQ three SDs above average has IQ $x = 3(16) + 100 =$ (choose one) **116** / **132** / **148**.

(m) A 20 year old with IQ two SDs below average has IQ $x = -2(20) + 120 =$ (choose one) **60** / **80** / **100**.

(n) Normal distributions **model** / **are actual** IQ scores.

(o) Random variable with normal distribution is **continuous** / **discrete**.

3. *Percentage, standard normal: temperatures.*
Temperature, Z, modeled as a *standard* normal, mean $\mu = 0^o$ and SD $\sigma = 1^o$.

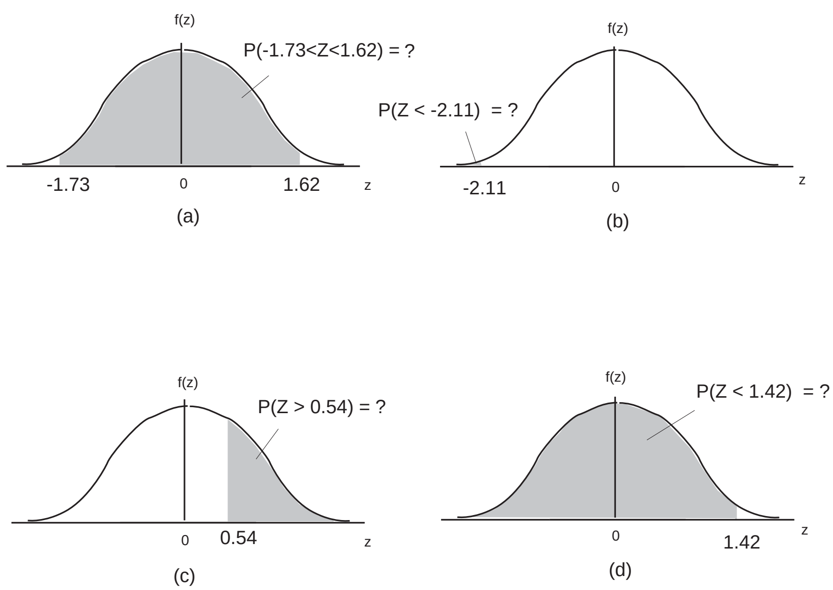

Figure 10.16 (Probabilities for standard normal)

(a) Probability temperature is between -1.73^o and 1.62^o ((a) above), $P(-1.73 < Z < 1.62) =$
0.0174 / **0.2946** / **0.9222** / **0.9056**.
(2nd DISTR 2:normalcdf(-1.73, 1.62,0,1).)[12].

[12]The normalcdf function has four arguments: normalcdf(low, high, ave, SD). In this case, "low" number is "− 2nd EE 99" and approximates negative infinity, "high" number is 1.42 and since this is a standard normal, ave and SD are 0 and 1, respectively.

(b) Probability temperature is less than -2.11^o ((b) above),
$P(Z < -2.11) =$
0.0174 / **0.2946** / **0.9222** / **0.9056**.
(2nd DISTR 2:normalcdf(− 2nd EE 99, −2.11,0,1).)[13]

(c) Probability temperature is greater than 0.54^o ((c) above),
$P(Z > 0.54) =$
0.0174 / **0.2946** / **0.9222** / **0.9056**.
(2nd DISTR 2:normalcdf(0.54, 2nd EE 99,0,1).)

(d) Probability temperature is less than 1.42^o ((d) above),
$P(Z < 1.42) =$
0.0174 / **0.2946** / **0.9222** / **0.9056**.

(e) **True** / **False** Probability temperature at *exactly* 1.42^o is *zero*.
Hint: Try 2nd DISTR 2:normalcdf(1.42, 1.42,0,1).

(f) **True** / **False** $P(Z < 1.42^o) = P(Z \leq 1.42^o)$.
Hint: Think about previous question.

4. *Percentiles, standard normal: temperatures.*

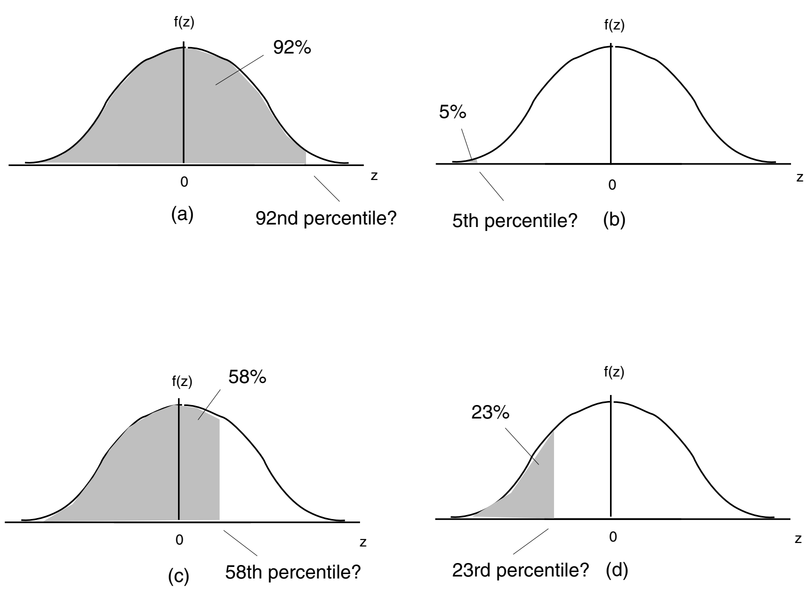

Figure 10.17 (Percentiles for standard normal)

(a) The 92nd percentile ((a) above) is
$\mathbf{-1.65^o}$ / $\mathbf{-0.74^o}$ / $\mathbf{0.20^o}$ / $\mathbf{1.41^o}$.
(2nd DISTR 3:invNorm(0.92,0,1)[14].

[13] "Low" number is "− 2nd EE 99" and approximates negative infinity; "high" number is −2.11.

[14] The invNorm function has three arguments: invNorm(percentile, ave, SD). "Percentile" rank is 0.92, not 92! And since this is a standard normal, ave and SD are 0 and 1, respectively.

(b) The z-score where area to left is 0.92 ((a) above) is
−1.65° / **−0.74°** / **0.20°** / **1.41°**.

Hint: same as 92nd percentile.

(c) If $P(Z < a) = 0.92$, $a =$ **−1.65°** / **−0.74°** / **0.20°** / **1.41°**.

Hint: same as 92nd percentile.

(d) The 5th percentile ((b) above) is
−1.65° / **−0.74°** / **0.20°** / **1.41°**.

(2nd DISTR 3:invNorm(0.05,0,1).)

(e) The z-score where area to left is 0.05 ((b) above) is
−1.65° / **−0.74°** / **0.20°** / **1.41°**.

Hint: same as 5th percentile.

(f) The 58th percentile ((c) above) is
−1.65° / **−0.74°** / **0.20°** / **1.41°**.

(2nd DISTR 3:invNorm(0.58,0,1).)

(g) The 23rd percentile ((d) above) is
−1.65° / **−0.74°** / **0.20°** / **1.41°**.

(2nd DISTR 3:invNorm(0.23,0,1).)

(h) Temperature, where 77% of temperatures are *above* this temperature is $z_{0.77} =$ **−1.65°** / **−0.74°** / **0.20°** / **1.41°** /

Remember 23rd percentile temperature has 23% of temperatures below it and 77% above it.

(i) Temperature, where 1% of temperatures are *above* this temperature is $z_{0.01} =$ **0.15°** / **0.74°** / **1.20°** / **2.33°** /

Remember 99th percentile temperature has 99% of temperatures below it and 1% above it.

(j) Since standard normal distribution is symmetric, centered at 0^o, and contains "100%" of the probability, 50th percentile (median)
below 0^o / **equal to** 0^o / **above** 0^o.

(k) **True** / **False** The 75th percentile is temperature with 75% of temperatures below it and 25% above it.

(l) Third quartile (75th percentile) is **below** 0^o / **equal to** 0^o / **above** 0^o.

5. *Percentages, normal: IQ scores.*

(a) For 16 year old IQ scores, where $\mu_{16} = 100$ and $\sigma_{16} = 16$,
$P(X < 84) =$ **0.8413** / **0.1587** / **−0.1587**

(2nd DISTR 2:normalcdf(− 2nd EE 99, 84, 100, 16)[15].

[15] The normalcdf function has four arguments: normalcdf(low, high, ave, SD). In this case, the "low" number is "− 2nd EE 99" and approximates negative infinity. The "high" number is 84. Also, this is a normal with ave and SD of 100 and 16, respectively.

(b) For 16 year old IQ scores, where $\mu_{16} = 100$ and $\sigma_{16} = 16$,
$P(X < 100) =$ **0.4413 / 0.5000 / 0.6587**
(2nd DISTR 2:normalcdf(− 2nd EE 99, 100, 100, 16).)

(c) For 16 year old IQ scores, where $\mu_{16} = 100$ and $\sigma_{16} = 16$,
$P(84 < X < 100) =$ **0.3413 / 0.4901 / 0.5587**
(2nd DISTR 2:normalcdf(84, 100, 100, 16).)

(d) For 20 year old IQ scores, where $\mu_{20} = 120$ and $\sigma_{20} = 20$,
$P(84 < X < 100) =$ **0.0413 / 0.1227 / 0.3597**
(2nd DISTR 2:normalcdf(84, 100, 120, 20).)

(e) *More probability questions.*

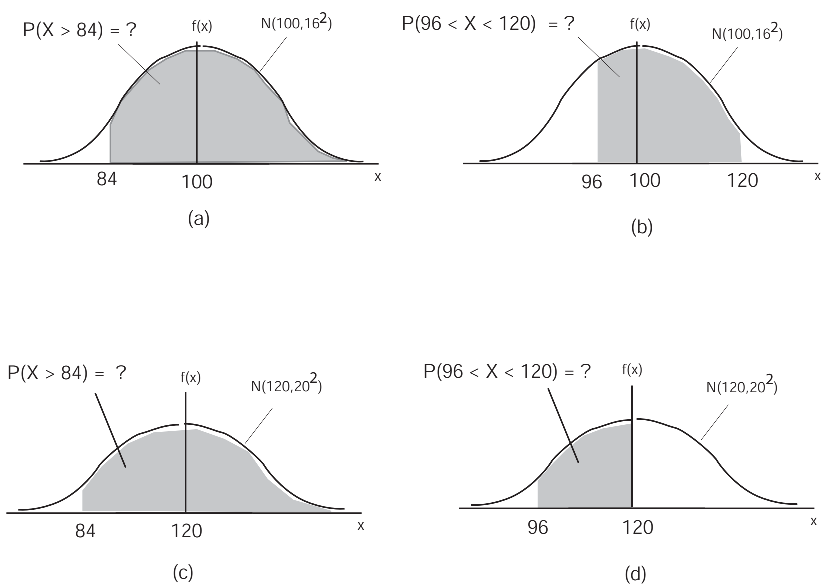

Figure 10.18 (Probabilities for normal)

Use calculator and figure above to match columns in following table.

Column I	Column II
(a) $P(X > 84)$, 16 year olds (2nd DISTR 2:normalcdf(84, 2nd EE 99, 100, 16))	**(a)** 0.4931
(b) $P(96 < X < 120)$, 16 year olds (2nd DISTR 2:normalcdf(96, 120, 100, 16))	**(b)** 0.9641
(c) $P(X > 84)$, 20 year olds (2nd DISTR 2:normalcdf(84, 2nd EE 99, 120, 20))	**(c)** 0.8413
(d) $P(96 < X < 120)$, 20 year olds (2nd DISTR 2:normalcdf(96, 120, 120, 20))	**(d)** 0.3849

Column I	(a)	(b)	(c)	(d)
Column II				

(f) Empirical rule says 68% of 16 year old IQ scores are inside interval $(\mu_{16} - \sigma_{16},\ \mu_{16} + \sigma_{16}) = (100 - 16, 100 + 16) =$ $\mathbf{(84, 116)}$ / $\mathbf{(68, 132)}$ / $\mathbf{(52, 148)}$.

(g) Empirical rule says 95% of 16 year old IQ scores are inside interval $(\mu_{16} - 2\sigma_{16},\ \mu_{16} + 2\sigma_{16}) =$ $\mathbf{(84, 116)}$ / $\mathbf{(68, 132)}$ / $\mathbf{(52, 148)}$.

(h) A 16 year old with IQ score 138 is **typical** / **unusual** because this score is greater than two SDs above average IQ score.

(i) Empirical rule says 95% of 20 year old IQ scores are inside interval $(\mu_{20} - 2\sigma_{20},\ \mu_{20} + 2\sigma_{20}) =$ $\mathbf{(100, 140)}$ / $\mathbf{(80, 160)}$ / $\mathbf{(60, 160)}$, so IQ score 138 is **typical** / **unusual** in this case.

6. *Percentiles, normal: IQ scores again.*

(a) 75th percentile for 16 year olds, where $\mu_{16} = 100$ and $\sigma_{20} = 16$, is: **103.5** / **106.7** / **110.8** / **112.3**.
(2nd DISTR 3:invNorm(0.75, 100, 16).)

(b) 32th percentile for 16 year olds, where $\mu_{16} = 100$ and $\sigma_{20} = 16$, is: **83.5** / **92.5** / **98.8** / **100.3**.
(2nd DISTR 3:invNorm(0.32, 100, 16).)

(c) 75th percentile for 20 year olds, where $\mu_{20} = 120$ and $\sigma_{20} = 20$, is: **133.5** / **106.5** / **125.4** / **142.3**.
(2nd DISTR 3:invNorm(0.75, 120, 20).)

(d) *More percentile questions.*

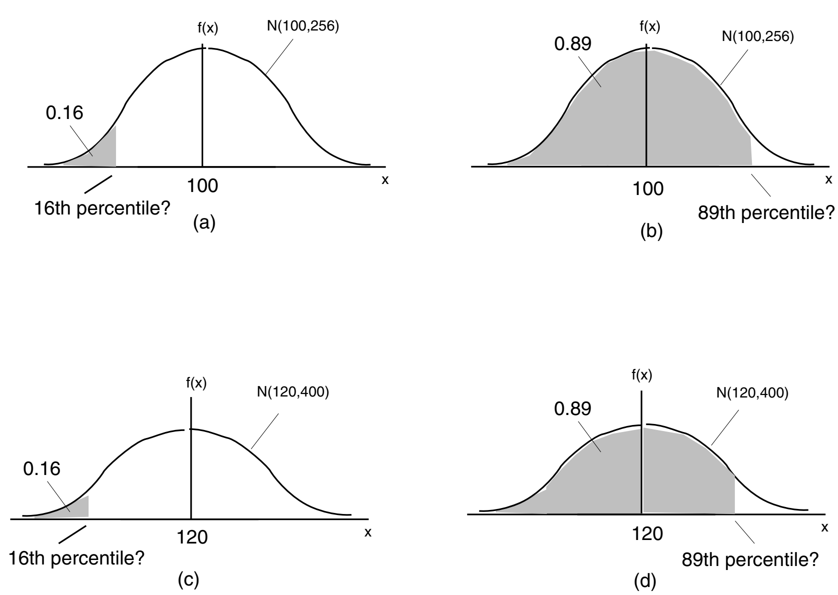

Figure 10.19 (Percentiles for normal.)

Use calculator and figure above to match columns in following table.

Column I	Column II
(a) 16th percentile, 16 year olds (2nd DISTR 3:invNorm(0.16, 100, 16))	**(a)** 119.6
(b) 89th percentile, 16 year olds (2nd DISTR 3:invNorm(0.89, 100, 16))	**(b)** 84.1
(c) 16th percentile, 20 year olds (2nd DISTR 3:invNorm(0.16, 120, 20))	**(c)** 144.5
(d) 89th percentile, 20 year olds (2nd DISTR 3:invNorm(0.89, 120, 20))	**(d)** 100.1

Column I	(a)	(b)	(c)	(d)
Column II				

(e) Since normal distribution of 16 year old scores symmetric, centered at 100, 50th percentile is **below** 100 / **equal to** 100 / **above** 100.

(f) 90th percentile, 16 year old scores:
below 100 / **equal to** 100 / **above** 100.

(g) 90th percentile, 20 year old scores:
below 120 / **equal to** 120 / **above** 120.

(h) **True** / **False**
90th percentile is IQ score with 90% of IQ scores below and so 10% above.

7. *More percentages and percentiles: chimp brain weights*
It was found in 1979 the brain weights of a certain population of adult chimps follow approximately a normal, mean 270 gm and standard deviation 40 gm.

(a) Percentage of adult chimps brains weighing between 250 gm and 300 gm: (choose closest one) **0.383** / **0.465** / **0.633** / **0.547** / **0.318**.
2nd DISTR normalcdf(250, 300, 270, 40).

(b) *Number* of 750 adult chimps brains weighing between 250 gm and 300 gm: (choose closest one) **249** / **301** / **349** / **397** / **403**.
Hint: $0.465 \times 750 = ?$

(c) The 80th percentile brain weight is:
(choose closest one) **304** / **308** / **310** / **312** / **314**.
2nd DISTR invNorm(0.8, 270, 40).

(d) A brain weight of 240 gm, expressed as a percentile, is
(choose closest one) ***12th*** / ***23rd*** / ***32nd*** / ***45th*** / ***55th***.
2nd DISTR normalcdf(− 2nd EE 99, 240, 270, 40).

8. *TI–84+: normal*

(a) *Probabilities and percentiles: standard normal and normal.*

- Assume the IQ scores for 16 year olds is normal where $\mu = 100$ and $\sigma = 16$. What is the probability a student randomly picked from the 16 year olds has an IQ score below 84?
- One way to do this would be to first *standardize* this probability, $P(X < 84) = P\left(Z < \frac{84-100}{16}\right) = P(Z < -1)$ and then use the "2ndDISTR/normalcdf" key to determine this probability:
  - 2nd DISTR 2 (-) 1 2nd EE 99, 84 , (-) 1 ENTER

  The value 0.1587 appears.
- Another way to do this is would be to calculate the probability $P(X < 84)$ directly. This also requires the use of the "2ndDISTR/normalcdf" key:
  - 2nd DISTR 2 (-) 1 2nd EE 99 , 100, 16) ENTER

  The value 0.1587 appears.
- Percentiles can also calculated. For example, in order to answer the question, "What is the 95th percentile for the 16 year old students?", use the "2ndDISTR/invNorm" key:
  - 2nd DISTR 3 ENTER .95 , 100 , 16) ENTER

  The value 126.32 is returned.

(b) *Graphing The Normal Distribution.* To graph the normal distribution with mean $\mu = -1$ and standard deviation $\sigma = 1$, type

- WINDOW −4 2 1 −0.2 0.6 0.1 1
- Y = 2nd DISTR 1:normalpdf(X , −1 , 1) GRAPH