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Chapter 3 Whole Numbers

3.1 Numeration Systems

Contrast other number systems with our own.

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The textbook may give three advantages but this supplement has four!

Numbers arose out of need – the imaginary numbers are a nice example.

Two models of our number system are presented but there are others such as the abacus.

Activities in other bases are valuable in illustrating the need and the usefulness of manipulatives.

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Base Ten or Dienes Blocks are very common in elementary schools.

Most mathematics elementary textbooks show pictures of Base Ten Blocks to illustrate numerical concepts.

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Reiterate: A manipulative is something that a child manipulates!

Elementary students often are going through the motions with expanded notation without understanding place value. Memorizing the places in a place value chart is not understanding place value. First children must learn the concept of ten and expand from there.

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Palindromes are interesting and hopefully enjoyable for the students.

The problem of changing 98 into a palindrome is a nice example requiring students to use paper and paper computation.

3.2 Addition and Subtraction

Addition is a counting activity for children. If we can understand how, specifically, children learn to add through counting, teachers can base their instruction on how children learn.

Eventually children start to learn facts such as doubles and use them to help them solve addition problems. Likewise they learn that subtraction is the opposite of addition and use their known addition facts to solve subtraction problems. But it is important to emphasize that when children are first learning to add and subtract they count.

One Child's Perspective of Addition

Hopefully, this example will generate some discussion and reflection.

Counting Types

Perceptual Counters

The most advanced thing perceptual counters can do is to use their fingers to form "collections." For 5 + 4, they will count "One, two, three, four five" as they put up five fingers on one hand, then "One, two, three, four" as they put up four fingers on the other hand. Then they will count all these fingers to determine the total. In the absence of physical materials, they will have great difficulty with sums over 10 and also with problems like 6 + 3—here they will typically put up six fingers for the first addend, but "lose" one of these when putting up three for the second addend.

Motor Counters

Verbal Counters

Summary and Discussion of Pre-Abstract Counting Types—Perceptual, Motor, Verbal

Emphasizing the memorization of basic facts at this stage diverts children from doing what makes sense to them (i.e., counting) and inhibits their progress through these levels of development. The problem is that perceptual, motor, and verbal counters have great difficulty memorizing the basic facts because numbers are not meaningful to them in the absence of objects that they can count. Asking these students to memorize the answers to 6 + 3, 2 + 8, 3 + 4, etc. is like asking them (or anyone) to memorize strings of symbols like

△ + ○ = □

▱ + □ = ▭

▭ + △ = ⏢

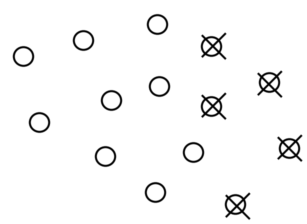
and then expecting them to regurgitate the answers (△ + ○ = _____) on a timed test.

Finger Counting

Should teachers allow children to use their fingers to solve addition and subtraction problems?

Part/Whole Counters
Another test for Part/Whole from Kamii is to put out 16 objects and write down the number 16. Ask the child what the 6 represents. Most children will circle or pull out 6 objects. Then the key question is, “What does the 1 represent?” A child who is not at the part/whole level will pull out 1 object.

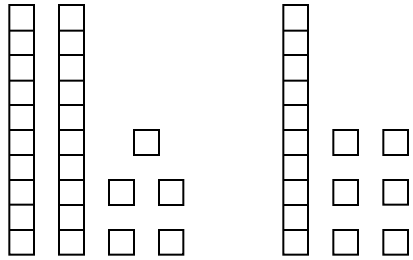
Note that being able to count back to solve a subtraction problem is conceptually different than being able to close the gap. In counting back, the child views 14 – 5 as having a collection of 14 and taking 5 away from it. The child does not see 5 as one part of the whole 14.



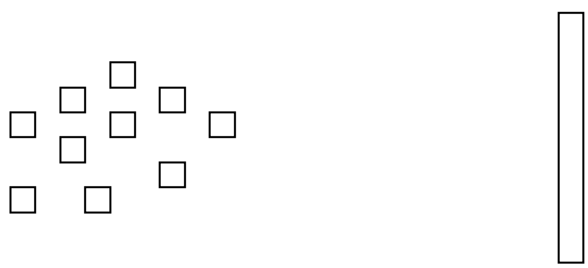
In this conception, there is no set–subset relationship as there is in the part/whole conception above. Of course, there is less counting involved in counting back to solve 14 – 5 than there is in closing the gap to solve it. Children at the part/whole level will still count back to solve subtraction problems when it is more efficient to do so. For example, given the problems 17 – 6 and 21 – 16, a part/whole counter would count back to solve the first one and close the gap to solve the second one. An abstract counter would count back for both of these problems.

<u>Concept of Ten and Place Value</u>

Children, who have not constructed the concept of ten, view 10 as ten separate units even when the problem (25 + 16) is presented with physical materials as in the figure below.



They have difficulty coordinating the counting of tens and ones because they do not see numbers as made up of tens and ones. That is, they do not think of 32 as three tens and 2 ones. Consequently, the only way they can solve problems like 32 + 10 or 57 + 10 is to count on by ones from 32 or 57, respectively. Presented with physical materials as in the above problem, a child at this level might count all the strips of ten first, "10, 20, 30," and then count the individual squares by ones. However, if the 16 was covered under a cloth and the child allowed to count the 25, told that 16 were hidden, and asked how many there were in all, the child would have to count on by ones from 25. Similarly, shown a collection of 24 (two strips of ten and four individual squares), told that some are hidden under the cloth and that altogether there are 57, a child at this level would count the 24 "10, 20, 21, 22, 23, 24," and then try to count on by ones from 24 up to 57 to determine the missing addend. Thus, it is as if children at this level see two separate units, big units (tens) and little units (ones), but they do not see them as related to each other. We could illustrate their two distinct <u>mental</u> representations of 10 at this level by the following figure.



10 as ten small units 10 as one big unit

Given a problem in which they are shown a collection of 37 squares (three strips of ten and seven individual squares), told that 24 are hidden under a cloth, and asked how many there are in all, children at this level might count "47, 57," putting up two fingers on one hand, and then "58, 59, 60, 61," putting up four fingers on the other hand. Or they might say something like "I took 3 from the 4 [hidden squares] and added them to the 7 to make another ten. That makes 4 tens, plus the two tens under the cloth makes 60. Then there's one [square] left over so 61." For the missing addend problem in which they are shown a collection of 24 (two strips of ten and four individual squares), told that some are hidden under the cloth and that altogether there are 57, children at this level might count "34, 44, 54," putting up three fingers on one hand, then count "55, 56, 57," putting up three fingers on the other hand, and conclude that the answer is 33. Or they might say "You need three more tens to make 50 and three more ones to make 7, so 33. After initial activities using these materials, children may be presented with problems using collections of strips of ten and individual squares (see above). Eventually, one of the collections may be covered and missing addend tasks may also be posed. These types of problems encourage children to build tens mentally.

The concept of ten is one of the most important concepts that we hope children learn by the end of second grade but not all third graders are part/whole.

Thinking Strategies and Learning "Basic Facts"

Five types of thinking strategies that children often use are the following:

1. Increasing Addend: +1 strategy (+2 strategy)
   Example: I know 5 + 5 = 10, so 5 + 6 = 11.
2. Decreasing Addend: −1 strategy (−2 strategy)
   Example: I know 5 + 5 = 10, so 5 + 4 = 9.
3. Compensation:

   Example: I know 5 + 5 = 10, so 6 + 4 = 10.
4. Inverse Relationship:

   Example: I know 7 + 5 = 12, so 12 – 7 = 5.
5. Filling Up Tens:

   Example: For 8 + 5 = ___, I took 2 from the five and added it to the 8 to make 10. Then I had 3 left over so I got 13.

Research indicates the learning the basic facts is not just a matter of simple memorization. Children need extensive experience solving single-digit addition and subtraction problems, first by counting, then by using thinking strategies, before they are ready to commit these facts to memory meaningfully. These initial problem-solving experiences help children build some understanding of numbers and of the operations of addition and subtraction. Once such understanding is developed, teachers may encourage children to begin to commit the facts to memory through some drill and practice activities. It is necessary for students to have the facts memorized or be able to figure them out very quickly in order for them to construct efficient methods for calculating with two- and three-digit numbers.

Two Levels of Difficulty for Subtraction

The textbook may make distinctions for addition, such as between adding measures and adding sets, but children typically do not make these distinctions. For children these types of problems are counting activities. This relates to the question of how mathematics instruction should be organized. Textbooks typically organize mathematics by the structure of mathematics. Others would advocate that mathematics instruction be organized based on how children naturally learn mathematics.

(Starts on page 33.) 3.3 Multiplication and Division

Third graders often say they multiplied but for them that means that they added and then used a multiplication number sentence to represent their thinking. For them multiplication is repeated addition.

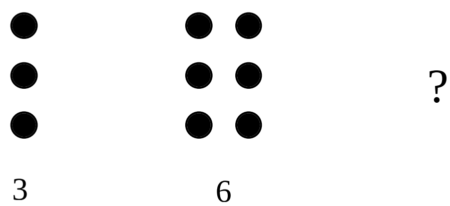
Multiplicative thinking grows out of additive thinking, but is more complex because it involves an additional level of abstraction (Piaget, 1983/1987). In an interview (Steffe, 1992), a third grader was asked to make four rows of 3 blocks each. The interviewer then covered these blocks and asked the child how many blocks were covered. The child correctly answered 12. Next, the interviewer asked the child to make two more rows like the other ones and to figure out how many blocks there were altogether. The child made the two additional rows, then, looking at the covered collection, said "4," and counted the blocks he had just added "5, 6, 7, 8, 9, 10." This is an example of a child at the second level in the development of the concept of 10 discussed above. The child could think in terms of units of 1 or in terms of units of 3, but could not coordinate these two ways of thinking. In order to think multiplicatively, children must be able to simultaneously think about units of one and units of more than one (Clark & Kamii, 1996). Children must be able to count units of 3, units of 4, etc. This does not just mean being able to count by 3's, 4's, etc., but being able to keep track of how many 3's, 4's, etc. they have counted.

Children pass through a series of levels of development in moving from additive to multiplicative thinking. A task that may be used to assess their development is the following, which is taken from Clark and Kamii (1996):

> A child is shown three "fish," 5, 10, and 15 inches long, respectively, and some plastic chips that will be used as "fish food." The fish differ only in length, not in any other dimension. The child is told that fish B (the 10-inch long fish) eats two times what fish A (the 5-inch long fish) eats and that fish C (the 15-inch long fish) eats three times what fish A eats. It is explained that fish B eats two times what fish A eats because it is two times as big as fish A, which can be demonstrated by showing that fish A could be placed on fish B two times. A similar explanation/demonstration is given for fish C and fish A. Then the child is presented with the question "If this fish (A) gets one chip of food, how many chips of food would you give to the other two fish? Remember that fish B eats two times what fish A eats and fish C eats three times what fish A eats." Subsequent questions might ask the child to determine how many chips to give to the other two fish in the following situations: when B receives 4 chips; when C receives 9 chips; when A receives 4 chips; when A receives 7 chips.

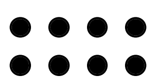
At the earliest level of development, children are able to think only qualitatively about this task. That is, they can think in terms of more or less, but will accept almost any answer as long as $A < B < C$. Such children have not yet constructed an abstract concept of number and do not think additively. At the next level, children's answers consist of +1 or +2 additive sequences. In other words, told that A receives 4 chips, they will give 5 to B and 6 to C (a +1 sequence) or they will give 6 to B and 8 to C (a +2 sequence). As their additive thinking develops, children attempt to accommodate the fact that B eats two times as much as A and C eats three times as much as A, but they add 2 and 3. In other words, told that A receives 4 chips, they will give 6 to B (+2) and 7 to C (+3). (Some may give 6 to B and 9 to C, adding the 3 to B's amount instead of A's.) At the next level of development, children display multiplicative thinking, justifying their correct answers to this task with talk of two groups of 4, three groups of 7, etc. (Clark & Kamii, 1996).

Activities involving dot pattern sequences can help with this.

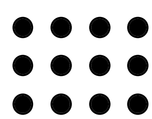


Can you keep this pattern going?

Additive thinkers also need activities that will encourage them to think in terms of units larger than 1. One such activity involves using an overhead projector to flash an array of dots (see below) on the screen for ≈ 3 seconds and asking, "How many did you see? How did you see them?"



Array 1



Array 2

By flashing these arrays on the overhead for a brief time, children are challenged to mentally organize the arrays in units larger than one because they do not have time to count them all by ones.

Problems that, from our adult perspective, might be called "division" problems can also help students think in terms of units larger than 1. Steffe and Killion (1989) identify three problems of this type. First, while students are not looking, the teacher may place, say, 12 tiles, glued together in pairs, under a paper plate. Then the teacher shows students a single tile and tells them that there are 12 such tiles under the plate. Showing the students a pair of tiles glued together, the teacher asks, "How many 2's do you think are under here?" A second problem involves giving students a large group of blocks, say 74, and posing the question: "There are 74 blocks in this group of blocks. Can you make stacks of ten?" After students make one stack, the teacher can ask, "How many stacks of ten could you make like that?" Finally, a third type of problem is one such as "If you count to 12 by 3's, how many 3's would you count?" This type of question can help students move beyond the need to physically make or see the units larger than one that they are counting.

Cartesian products are not a good way to introduce multiplication to children because this is not initially how they naturally think about multiplication. However, Cartesian products are useful later in the study of probability.

Introducing Two-Digit Multiplication

The two-digit multiplication illustration can also be used to illustrate multiplication of decimals (tenths) and multiplication of fractions.

Another nice illustration of multiplication is lattice multiplication.

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<u>How Young Children Divide</u>

Ask students how they thought young children did division before reading this section?

Homework Assignment

Have students create 8 word problems, appropriate for second graders in both number and content. Problems in which the computation is something like 1 + 2 are too easy. Problems with double or even triple digits are more appropriate. Numbers in the millions are much too large. Students should try not to use key words!! To illustrate why key or cue words should not be used, consider the following problem: If Johnny walked 5 blocks, then turned left and walked 3 blocks, how many blocks did he walk? Many children, especially those that have been taught key words, will subtract (5 − 3) because they saw the word "left." We want children to think about the problems given to them, not just look for two numbers and an operation. Ask your students to make up 1 problem that involves both addition and subtraction, 2 take-away problems, 2 compare problems, 1 multiplication problem, 1 division problem where the solution method used is partitioning, and 1 division problem where the solution method used is a repeated operation. They should identify which of their problems correspond to each of these categories. Students are to solve the multiplication problem by repeated addition and the two division problems by the method given.

Division By and With 0

Ask students for the solutions to these two problems. They often know that one is 0 and the other cannot be done, but get the two mixed up. Ask them, “What do you remember your teacher telling you about these problems?” As future teachers they should be able to explain to children why division by 0 is undefined and why 0 divided by a natural number is 0.

You might draw arrows to show how each division is changed the same way into a multiplication problem.

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Order of Operations

The order of operations is a convention—that is, there is not a mathematical justification why, for example, multiplication is done before addition. Hopefully, students can see the need for these rules and can explain why they are needed to children.

3.4 Properties of Whole-Number Operations

Commutative Property

To illustrate how children think about the commutative property of addition, try the following activity, called "Target," with your class. In the activity of Target students are to give two numbers that give the target number—in this version they may only use one operation, addition. Suggest that they pretend that they are first or second graders. Start with the target number "10." Write it on the board. Ask students for pairs of numbers whose sum is 10.

Invariably, you will get the following pairs of numbers: 5,5; 6,4; 7,3; 8,2; 9,1; and 10,0. However, first and second graders never stop here, unless they have been directly told to do so. They normally go on to give the following pairs: 4,6; 3,7; 2,8; 1,9; and 0,10. Ask your students, "Why don't children stop?"

Other students may use physical materials to represent the addition of two quantities. Interestingly, the use of physical materials can actually promote the construction of a more abstract argument because it enables students to make generalizations about how the addends and the sum are related to each other under the operation of addition. Still other students may explore whether or not the notion of "turn-arounds" holds for subtraction. This can lead to ideas about negative numbers (Schifter, 1997).

Multiplication

This is as it should be because the teacher needs to negotiate these conventions for interpreting 4 × 3 and 3 × 4 with the class in order to facilitate communication. (If some students are viewing 4 × 3 as four 3's and others are viewing it as three 4's, attempts at mathematical discussions are likely to degenerate into arguments about how to interpret 4 × 3. So it is necessary to establish with the class exactly how this will be interpreted.) In the case of addition, children may reason that 8 + 5 is the same as 5 + 8 because you could take 3 from the 8 and put it with the 5, thereby transforming the one number sentence into the other without altering the sum at all.

Associative Property

Begin by writing the first part, 17 x 25. Ask students to do this problem mentally. They typically cringe until you add the second part. Then most can see an easier way.

Closure Property

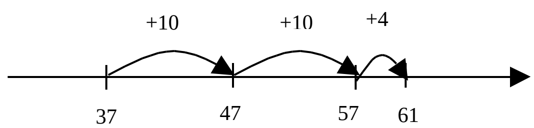
Emphasize the closure property from a mathematical standpoint. It can be considered the driving force for the creation of numbers.

Mathematically, addition is closed for whole numbers but subtraction is the driving force for the creation of integers. Likewise, multiplication is closed for whole numbers but division is going to cause us to create rational numbers or fractions!

3.5 Algorithms

Students should know what the standard algorithms are so that when they hear the term “standard algorithm” they know what is being referred to.

For, 37 + 24, such a strategy is called a linear-based strategy because it could be represented on a number line as follows:



A different type of algorithm children often apply to such a problem might go like this: “30 + 20 = 50, and 7 + 4 = 11, take 10 from the 11 and add it to the 50 to make 60, then there’s 1 left over so the answer is 61.” This type of strategy is called a collections-based strategy because it involves thinking about numbers as collections of tens and ones. For the subtraction problem 72 − 24, a linear-based strategy that children employ is “72 − 10 = 62, 62 − 10 = 52” (or “72 − 20 = 52”), then “52 − 4 = 48.” A collections-based strategy for this problem is “70 − 20 = 50, 50 – 4 = 46, 46 + 2 = 48” or “take 10 from the 70 and add it to the 2 to get 12, then 60 − 40 = 20 and 12 − 4 = 8, so the answer is 48.”

In closing this section, we present a second-grade teacher’s description of and reflections on her students’ self-generated subtraction algorithms.

> Fiona worked on a variation of the word problem that involved regrouping (of 37 pigeons, 19 flew away). She dropped the 7 from the 37 for the time being. She then subtracted 10 from 30. Then she subtracted 9 more. She puzzled for a while about what to do with the 7, now that she had to put it back somewhere. Should she subtract it or add it? I asked her one question: Did those seven pigeons leave or stay? She said they stayed, and added the 7.
>
> 37 − 19
>
> 30 − 10 = 20
>
> 20 − 9 = 11
>
> 11 + 7 = 18
>
> It was interesting to me that Fiona was able to use that one question to clear up her confusion, and I think for the most part she subtracts this way and keeps it straight. As Fiona goes through the steps in her algorithm she is able to keep track of when to add and when to subtract. The 7 is being subtracted (from 37) and then added again (at the end, to the 11). The 9 from the 19 is in a way added to the 10 in 19, but it gets subtracted, because Fiona needs to subtract all of the

19. The 7 is part of what is being subtracted from. The 9 is part of what is being subtracted. It is a complicated process and it is amazing to me that a second grader can make sense of it for herself. . . .

Paul also takes numbers apart to subtract. To solve 39 − 17, he takes the 17 apart:

$$39 - 10 = 29$$

$$29 - 4 = 25$$

$$25 - 3 = 22$$

Paul is keeping track of the 17 and breaking it into familiar chunks. Many children wondered where he got the 10, 4, and 3 from. How did he know what to subtract? How did he know when he was done?

Interestingly, Paul had questions for Nathan about how Nathan knew which numbers to put together for his answer. Here is Nathan's process for 39 − 17:

$$17 + 3 = 20$$

$$20 + 10 = 30$$

$$30 + 9 = 39$$

$$3 + 10 + 9 = 22$$

Nathan just about always adds, even for what seems like a straightforward separating situation like birds flying away. After Nathan told how he solved this problem, Paul said, "But how does he know what numbers to add up at the end?"

I thought a little bit about children using the conventional algorithm. A few do sometimes, ever since I gave word problems for homework. So much for asking parents not to help them. If a child memorizes the procedure, there is no real "keeping track." They must learn the steps, but they do not need to keep track of what the 3 in 37 means or how much of the 19 they have subtracted so far. All they do is use the recipe. If they get confused or forget a step or go out of order, children using this procedure don't tend to go back and make sense of the numbers in the problem, or try to keep track of what is going on.

Finally, an almost unrelated observation: This year, for the first time, I have never seen a single child "subtract up" in the ones column if the bottom number is greater than the top one. I have always had many children do this other years.

$$\begin{array}{r} 37 \\ -\underline{19} \\ 22 \end{array} \qquad \text{because } 3 - 1 = 2 \text{ and } 9 - 7 = 2$$

I am not sure what to make of this, but I hope it is because the children this year carry more of the meaning of the problem with them, because they are allowed to construct their own ways of solving it. (Schifter, 1997)

<u>Multiplication Algorithms</u>

Present the students with an example of the standard multiplication algorithm like that below.

$$\begin{array}{r} {}^{2} \\ {}^{2} \\ 47 \\ \times\ 34 \\ \hline 188 \\ 1410 \\ \hline 1598 \end{array}$$

Working with a partner or in a group, ask them to try to explain and justify why this standard algorithm works. They should not just say what the steps are—they should explain the mathematical significance of each step. Some particular questions you might ask are "Why do we always put a 0 for the right-most digit in the second line underneath the top horizontal bar?"; "What do the 2's above the 4 in 47 mean?"; "Why don't we add the 2 to the 4 <u>before</u> multiplying?"; "Why do we add 188 and 1410? Why not multiply?"

You might also ask students to find an alternate way to compute 47 × 34.

Below we present a third-grade teacher's account of some of her students' initial efforts to solve multi-digit multiplication problems. The account illustrates the important role that an understanding of place value and the ability to partition numbers in different ways play in the development of multiplicative thinking. The teacher had presented the class with the following problem: "There were 64 teams at the beginning of the NCAA basketball tournament. With 5 players starting on each team, how many starting players were in the tournament?"

> "Wow, that's hard," proclaimed Jenny loudly, and a chorus of protesters joined her. Undaunted, Julia presented her thinking:
>
> > That would be 64 × 5. I use one 10, because I know 5 × 10 = 50. Then you do that six times. (She counted by 5s, not using her fingers, but moving her lips and nodding her head for each group of 5.) That's 30, I mean 300. Then you add 4 five times, which is 25, no 20. I added it all together and got 320.
> >
> > $$\begin{array}{l} 5 \times 10 = 50 \\ 5 \times 10 = 50 \\ 5 \times 10 = 50 \\ 5 \times 10 = 50 \\ 5 \times 10 = 50 \\ 5 \times 10 = \underline{\ 50\ } \end{array}$$

300

4 + 4 + 4 + 4 = 20

300 + 20 = 320

Chris, usually reticent and lacking in confidence, volunteered his thinking in a quiet, unassuming voice:

> 64 means 60 + 4. . . . So I did 60 five times, for 300. Then 4 × 5 is 20, so the answer is 320.

60 × 5 = 300

4 × 5 = 20

320

Chris returned to his seat in a way I can only describe as cocky. I was certainly impressed. Jack, our resident goof-off, but intuitive math thinker, explained his strategy next:

> I split the 64 into four parts—20, 20, and 20. . . . I did each one separately.

20 × 5 = 100
20 × 5 = 100
20 × 5 = 100

> Then the last part, 4 × 5, is 20. All together, 320.

These were the ideas and strategies I'd tried so hard to explain and instill in my students last year—breaking up numbers in useful parts, recognizing which numbers are being multiplied and by how much, finding a way to multiply that makes sense. Posing the right questions and relying on the children to use what they'd been practicing all year proved to be the solution to teaching multiplication. Of course, not everyone had moved beyond adding, or understood what their colleagues were doing. But they were listening and would begin to develop new strategies as we continued to multiply.

. . . I was just about to hand out colored tiles for building arrays, when another multiplication opportunity presented itself.

> Teacher: We have 18 kids here today and each needs 12 tiles for the next activity. How can we figure out the number of tiles to give out?

It took me a second to realize this problem was a leap from the ones we'd just done. But it was "real life," so I let the question stand.

I was surprised that no one suggested using a calculator, their usual response to big numbers. But who needs a calculator when you have Josh!

> That would be 18 × 12, and I know 10 × 10 is 100 and 8 × 2 is 16, so if you add them together it would be 100 + 16 = 116.

Everyone seemed satisfied with the answer, whether out of agreement or lack of interest, I wasn't sure. After all, the process mimicked what they'd just been doing. I was thinking what to say that would help them see the error of their ways, when David's voice broke the quiet: "That's wrong."

"What do you mean, David?" I asked.

> I did 18 × 10 and got 180, but I thought at first I was wrong, so I double-checked. I noticed that Josh didn't do 8 × 10, so my answer was right. (David is very knowledgeable about the workings of our number system, but leaves gaps in his verbal explanations. His mind races, and neither his mouth nor our brains can keep up.) I didn't do the 2 yet, so I do 18 × 2. Then you add it up—180 + 36.

"Wow," I thought, amazed at his understanding, but realizing that the rest of the class looked dazed. Luckily, there will be more chances for [these children] to show what they know about multiplication. (Schifter, 1997)

As the last portion of this episode illustrates, the transition from multiplying a two-digit number and a one-digit number to multiplying two two-digit numbers is far from trivial. Josh attempted to adapt the process the class had used to find 64 × 5 by employing a strategy he was familiar with from addition, namely breaking the numbers into tens and ones, computing with the tens, computing with the ones, and then putting everything back together. However, David pointed out that this method missed the 8 × 10 that was included in his 18 × 10. (Of course, it also missed the 10 × 2. Think about an 18 by 12 array like the one shown previously.)

Implicit in all the solution methods presented in the above episode is the distributive property of multiplication. For example, we could describe Chris' solution as follows:

$$64 \times 5 = (60 + 4) \times 5 = (60 \times 5) + (4 \times 5).$$

Likewise, Josh's solution claimed that

$$18 \times 12 = (10 + 8) \times (10 + 2) = (10 \times 10) + (8 \times 2).$$

However, David argued that Josh's solution was incomplete. Instead,

$$18 \times 12 = 18 \times (10 + 2) = (18 \times 10) + (18 \times 2).$$

This type of understanding of how multiplication works is a critical component in children's development of efficient algorithms for multi-digit multiplication.

As the teacher noted in closing her account, most of her students will need many more opportunities to apply their developing strategies for multiplication in order to reach David's level of understanding and beyond. But by providing these opportunities through engagement in the types of activities and discussions described in the account, the teacher is enabling students to make sense out of multiplication by connecting it to what they already know. By undertaking discussions in which the students' understanding of their work is the heart of the issue, the teacher is helping them construct ideas that will support their continued learning. In fact, discussions like the one reported here may be important not only in the immediate context of learning about multiplication, but may also have consequences in the later study of algebra. Schifter (1997) says

> When students have learned arithmetic as a set of memorized procedures and have lost contact with their own abilities to make sense of calculations and operations, it is no wonder they have to rely on remembered rules and procedures to pass an algebra course. For example, what if Josh … never had confronted his error in solving 18 × 12 by calculating 10 × 10 + 8 × 2? What if there were no occasion in his elementary education to think through what multiplication does so that he comes to understand why 18 × 12 *can* be solved by calculating, instead, 18 × 10 + 18 × 2? If he someday enters algebra class without that understanding, how is he to learn how to multiply binomials? What else can he do to learn that $(a+b)(c+d)$ does not equal $ac+bd$, but memorize the correct identity? … When Josh and his classmates eventually confront "$a(b+c)=ab+ac$," they will bring to it their discussions of why 18 × 12 is equal to 18 × 10 + 18 × 2, rather than 10 × 10 + 8 × 2. (pp. 20-21)

Division Algorithms

Present the students with an example of the standard long division algorithm like that below.

```
     164
26 )4283
    26
    168
    156
     123
     104
      19
```

Working with a partner or in a group, ask them to try to explain and justify why this standard algorithm works. They should not just say what the steps are—they should explain the mathematical significance of each step. Some particular questions you might ask are "What does the 26 in the first line underneath 4283 mean?"; "We begin by asking

'How many times does 26 go into 42?' but what are we really finding out?"; "When we perform the first subtraction, why do we bring the 8 down but not the 3?"

You might also ask students to find an alternate way to compute 4283 ÷ 26.

Gravemeijer (1994) presents two such problems and illustrates a variety of solution strategies students have been known to employ to solve them:

> The captain of a stranded ship is told that there are 4000 biscuits left. The crew consists of 64 members. Each man gets 3 biscuits a day, which means 192 biscuits a day for the whole crew. How long will this supply last?

4000		4000		4000	
— 192	1 day	— 192	1 day	— 1920	10 days
3808		3808		2080	
— 192	1 day	— 384	2 days	— 1920	10 days
3616		3424		160	
— 192	1 day	— 768	4 days		
3424		2656			
— 192	1 day	— 1536	8 days		
etc.		etc.			

> 1296 fans want to visit an away soccer game of their favorite team. The treasurer of the fan club learns that one bus can carry 38 passengers and that a reduction in price will be given for every 10 buses the club books. How many buses should the club book?

1296		1296		1296	
— 380	10 buses	— 380	10 buses	— 1140	30 buses
916		916		156	
— 380	10 buses	— 760	20 buses	— 152	4 buses
536		156		4	
— 380	10 buses	— 76	2 buses		
156		80			
— 38	1 bus	— 76	2 buses		
118		4			
— 38	1 bus				
80					
— 38	1 bus				
42					
— 38	1 bus				
4					

Note that the mention of a price reduction for every ten buses is intended to encourage students to make use of multiples of 10 and thus develop an efficient repeated subtraction algorithm. While the two problems above fit into the measurement model of division, problems based on the fair-sharing model can also facilitate construction of division algorithms. Faced with the problem of distributing 231 M&M's into 5 containers, students develop strategies like that below:

1	1	1	1	1
5	5	5	5	5
10	10	10	10	10
20	20	20	20	20
10	10	10	10	10

231 – 50 = 181, 181 – 100 = 81, 81 – 50 = 31, 31 – 25 = 6, 6 – 5 =1
46 in each bin with 1 left over

In other words, the student decides to put 10 M&M's in each bin first, calculates the number remaining (i.e., 231 – 50), realizes that a larger collection than 10 could still be put in each bin so allocates 20 M&M's to each bin on the second pass, calculates the number remaining (181 – 100), etc., continuing in this manner until fewer than 5 M&M's remain.

?" A more accessible alternative to the standard algorithm may be drawn from students' self-generated algorithms. This slightly expanded, yet still efficient, algorithm is illustrated below for the fan bus and M&M problems.

38 ⟌1296	
760	20
536	
380	10
156	
76	2
80	
76	+ 2
4	34

1	
5	
10	46
20	
10	
5 ⟌231	
50	
181	
100	
81	
50	
31	
25	
6	

5
1

This alternative algorithm may be viewed in terms of either model of division (the fan bus problem used the measurement model, the M&M problem the fair-sharing model), does not require exact multiples of the divisor (thus breakdowns other than those shown above are possible for each problem), and retains the place value meaning of all numbers involved.

3.6 Mental Math & Estimation

Mental math and estimation are important skills that children will use in their daily lives. Unfortunately, these types of activities in elementary mathematics textbooks are not realistic to children. Children learn to round, but in real life there are times that you round down and others that you round up without regard to the rules of rounding. The homework assignment provided attempts to address this, but if students consider it a textbook exercise, then it also misses its intended purpose.

Of course, the traditional standard algorithm may be applied to division problems corresponding to either model, but, as noted, the typical language of “How many times does 5 go into 231?” corresponds to the measurement model. With the alternative algorithm, one might think, “How many times does 5 go into 231?” but one might also think, “If I’m distributing 231 among 5 groups, how many could I put in each group?”

Instructor Supplement

Chapter 4 Number Theory

Number theory is an interesting topic and in many cases the mathematical ideas are accessible to preservice teachers.

4.1 Factors and Multiples

The notation of divides, 6ê42, can be new and confusing to students.

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Typically, teachers will not be teaching that 0 is a multiple of every number to children.

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4.2 Divisibility Tests

Solve: 6½23, 45_

This is a good problem because it is a good indication whether or not college students understand the tests. However, they could use trial and error to solve it as well.

4.3 Prime and Composite Numbers

You may want to explain how to construct a Sieve of Eratosthenes even if it is presented or assigned in the textbook.

You may want to let them use the sieve on tests and the final.

The true/false questions can be useful in the discussion of what constitutes a proof*. For children and college students, several examples usually make something true in their minds.

True or False: All prime numbers are odd.

This is a nice example because, even though there are infinitely many odd primes, the statement is false because there is one even prime.

*Although this chapter is about number theory, it is important to connect mathematical ideas that elementary teachers will encounter whenever possible, such as the concept of proof.

4.4 Greatest Common Factor & Least Common Multiple

"Relatively prime" may confuse students but it is a term that may be used on tests and helps convey an important mathematical idea. For example, if the numerator and the denominator of a fraction are relatively prime, 4/9; GCF (4,9) = 1, then the fraction is in lowest terms.

In elementary school children will be making lists to find the GCF and LCM.

LCM = LCD

Prime factorization is typically taught in middle school.

To address the concepts of factors and multiples in the Locker Problem you could also ask: Which students touch the 42^{nd} locker? And Which lockers do the 4^{th} and 7^{th} person both touch?

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