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CMET
Connecting Mathematics for Elementary Teachers
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Chapter 1 Problem Solving

1.1 An Introduction to Problem Solving

What is Problem Solving?

All elementary mathematics textbooks contain sections on problem solving and some suggest a problem-solving approach to teaching all topics.
Encourage students to check out the website describing the reform-oriented curricula!

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Learning to Teach Problem Solving

Preservice teachers are doing problem solving because it will help them learn to teach problem solving. Many have not experienced either a problem-solving approach or even general problem solving in their learning of mathematics.

What is a Problem?

There is no problem without a person. A problem to one person may be a routine exercise to another. You cannot give problems to children. Teachers should attempt to make sure that the tasks presented really are problems for students. Activities should be developmentally appropriate—not too difficult and not too easy! Just as important, even if the teacher has selected appropriate problems, he or she must try to establish that students share the intended goal, which is to solve and understand the problems and their solution methods.

Problem solving should be challenging for preservice teachers as well. It is okay for them to struggle with problems, but try to avoid creating anxiety. It really is fun to solve challenging problems.

A Mathematical Problem for Children

Try this activity with your class if you are going to be doing group work.

A viable solution for first and second graders is: if your number ends in 1, 3, 5, 7, or 9, then your partner is the next higher number; if your number ends in 0, 2, 4, 6, or 8, then your partner is the next lower number.

How Children Solve Problems

Research indicates that children are much more likely to act out or model problems than are adults. Adults tend to try to solve problems mentally.

Problem # 4 in the **Problems – Focusing on the Process** section is a nice example of how acting out the problem can help because if you put four books on a shelf it is more obvious where the first and last page of each book is.

Often a first step in problem solving is to do something even if it is incorrect. However, that first step often gets the student going toward a viable solution.

Problem Solving Is Not Just Word Problems

Problems 1, 7, and 8 from the **Problems – Focusing on the Process** section can be used to illustrate this point.

Problems - Focusing on the Process

These problems should not all be assigned at once. You might spend an entire class session on problem #1.

Focus on the process!

The rationale for doing problem solving is that it may help students construct mathematical relationships and it will encourage them to develop their own strategies and processes for solving problems. However, traditional textbooks' approach to problem solving often fails to capture its essence because children are not engaged in an inquiry process on mathematically challenging activities. Wheatley characterizes many textbooks' approach to teaching problem solving as "the solving of well-defined questions based on certain information provided, frequently with the method specified" (Wheatley, 1999).

1. Find the sum: 1 + 2 + 3 + ... + 998 + 999 + 1,000 = ?

Students will discover some interesting patterns. For example, some may find a pattern by adding in groups of ten:
1 + 2 + 3 + 4 + 5 + 6 + 7 + 8 + 9 + 10 = 55
11 + 12 + 13 + 14 + 15 + 16 + 17 + 18 + 19 + 20 = 155
21 + 22 + 23 + 24 + 25 + 26 + 27 + 28 + 29 + 30 = 255
31 + ... + 40 = 355
Others may find a similar pattern:
1 + 2 + 3 + 4 + 5 + 6 + 7 + 8 + 9 + = 45
10 + 11 + 12 + 13 + 14 + 15 + 16 + 17 + 18 + 19 = 145
Then they face the task of adding the pattern of sums. Some may simply chug it out while others will look for other patterns to add the pattern of sums. Students are frequently impressed with each other's thinking, indicating that they never knew that there were so many different strategies to solve problems. Often they take unviable strategies to solve the problems but, in the long run, they appear to have a better understanding of mathematical processes. For example, typically a few groups will say that 9,955 is the answer because they added the last ten numbers of the sequence:
991 + 992 + 993 + 994 + 995 + 996 + 997 + 998 + 999 + 1,000 = 9,955
After they have explained this solution, you might ask if they have added all the numbers in the sequence.

This is a common problem in mathematics content courses for elementary teachers. Many, if not most, textbooks will suggest using Gauss's method or formula for this problem. However most students do not understand this process. Instead, after they try the problem on their own, you could look at Gauss's method and perhaps ask the students to use it to find 1 + 2 + 3 + … + 998 + 999 + 1000. This may help them appreciate the ingenuity and efficiency of this method.

Multiple strategies

Certain problems lend themselves to different discussions, which are highlighted below, but which may not always occur depending on your students.

2. A protractor and a compass cost $3.00. If the protractor costs $.80 more than the compass, how much does each cost?

Some students use algebra, but you may not even want to go over the algebra solution. Just accept it and tell students that if they understand algebra they are free to use it, but elementary school students will never use this solution method.

Others subtract .80 from 3.00 and divide the result by 2 to get one of the solutions. Once students have given this solution you can use your hands to demonstrate why this solution works. Hold up your hands at different heights, representing the values of the two items. If we subtract the difference between the two from the larger value, then they are equal. (Bring the higher hand down to be with the lower one.) Since we know the total of these two equivalent values (3.00 – 0.80), we can divide it in half. (Split your two hands apart.) Now we have one of the values.

Another solution is to divide the 3.00 in half and the .80 in half and then add .40 to 1.50 and subtract .40 from 1.50. You can model this by again holding up both hands, one higher than the other. If we average the difference between the items and then bring the higher one half way down and the lower one half way up, they will be equal.

Occasionally students may also use trial and error. For example, they might start with 1.50 and assume that the other item is 2.30 and keep adjusting these numbers until they reach 3.00. Elementary students may be more likely to use this method than your college students.

3. Looking in my backyard one day I saw some boys and dogs. I counted 24 heads and 72 feet. How many boys and how many dogs were in my backyard?

Algebra is again one method, as is trial and error, but a neat method that elementary students might use involves drawing a picture of the situation. They will draw out 24 heads and then give each head 2 feet. Then 72 – (24x2) = 24 feet left over. Next they distribute the 24 feet 2 at a time to the heads.

Mention to your students that children are likely to use this method and it is perfectly acceptable for them to do the same—it is a valuable problem-solving strategy.

4. There are four volumes of Charles Dickens's collected works on a shelf. The volumes are in order from left to right. The pages of each volume are exactly 2 inches thick. The covers of each volume are exactly 1/6 inch thick. A bookworm started eating at page 1 of Volume I and ate to the last page of Volume IV. What is the distance the bookworm traveled?

If the class has not figured it out, hold up a book and ask them to think about where page 1 of the book is when it is on a shelf. This is a great problem to encourage students to think.

5. How many fence posts will it take to fence a rectangular field 250 feet by 300 feet if each fence post is exactly 5 feet apart?

Many students divide each side by 5 to find the fence posts on each side; then they decide that they must subtract 4 for the four corners. However, sometimes students look for too much because you do not need to subtract 4. You could think of it as stretching the fence into a straight line. If the class is in disagreement over this problem, you might suggest that they apply the strategy of breaking the problem into smaller parts as a way to help them work out the disagreement.

6. If a snail is at the bottom of a well that is 100 feet deep and he climbs up 8 feet each day but slips back 5 feet each night, how long will it take him to climb out of the well?

Many students divide 100 by 3 to get 33.3 and then round up and indicate that he will get out in 34 days. A more illustrative strategy is to break the problem into smaller parts and look near the top, say 90 feet, which would take 30 days, and work it out from there.

7. Which number does not belong?

   15 23 20 25

It is possible to find more than one reason why each of the numbers does not belong! One of the more interesting solutions is to add the digits and see which one does not fall in the possible sequence!

Students often believe that there is only one correct answer to a problem so discussing the problem above with the class may help to challenge that belief.

8. How many rectangles are in this figure?

Children and college students frequently begin by tracing rectangles but this method often leads students to come up with different answers. Begin the discussion by asking different students how many rectangles they found. Record their answers on the board. Then ask the different students to explain their strategies, beginning with the student who reported the smallest number and working your way up. If you take an organized approach, such as listing the number of 1x1, 1x2, 1x3, 1x4, and 1x5 rectangles, they all fall into a pattern. The number of 2x1, 2x2, 2x3, 2x4, and 2x5 rectangles fall into a pattern as well. Also, if you increase the figure to 2x6 the same patterns occur again.

In fifth grade a rectangle is defined as a quadrilateral with four right angles. Ask your students how kindergarten and first-grade students might describe a rectangle.

Problem Solving Steps

Research is beginning to show that the four-step approach is not how mathematicians and scientists really think about a problem (Safford, 1994). They say that they rarely understand a problem until after it is solved. And their “plan” could not be called a real plan but perhaps a hunch or intuition. Experts become better problem solvers by solving many problems and have a repertoire of strategies or techniques that they can fall back on.

It is okay for children to struggle with problems. Frequently, teachers feel that as soon as a child is struggling they should jump in and tell the child what to do. This has been what they consider “good teaching.” But does it make for good learning?

Problem Solving Strategies – Heuristics

One of the most frustrating aspects of using a textbook to teach problem solving can be the book's treatment of heuristics as rules. Wheatley states: "Heuristics are not Rules!" (Wheatley, 1984). Instead, they are tools that people often use when solving problems. These tools help them understand the problem and organize their thinking about it. Yet, often in textbooks a heuristic is given or suggested for each word problem (i.e., exercise). Students end up applying known procedures to tasks without thinking about the task. This is not problem solving.

To counteract this methodology, take the best problems, mix them up, and present them to students, omitting the suggested heuristic. Initially, do not discuss heuristics or allow students to look in the textbook. Students would probably solve the problems more quickly if they used the textbook and the suggested heuristic, but they would never develop many of the strategies and thinking processes that this more open approach enables them to construct for themselves. For a given problem, if the textbook suggests a heuristic like "guess and test," that will be the only strategy that students try even though other strategies are also appropriate. In solving real-life problems, no one is there to tell students what heuristics to use.

After students have had an opportunity to solve problems, usually over the course of several days, stop and talk about heuristics and how the textbook presents the same problems. With the approach described here, students appear to have a good understanding of problem-solving strategies, they are open to various methods of solving problems, and they are not afraid to attempt unfamiliar problems. These students contrast with students who have been directly taught heuristics—students who typically are stuck when they encounter a problem that does not match a heuristic they have studied. Note that the approach to problem solving described above may also be used with elementary school students.

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Heuristics can be valuable tools for problem solving, but they are only tools. Students will choose the tools that they are most comfortable with and that make sense to them. Different carpenters may choose different tools to complete the same job, but they both get the job done. The better the tools that one has at his or her disposal, the better and more efficient one can be at completing the job. Hence, our goal as teachers of problem solving is to provide opportunities for most students to learn and feel comfortable with a variety of tools. The goal is not always the answer to the problem!!!!

1.2 Patterns

Some key points/questions related to this section are the following:

- Is mathematics created or discovered?
- New textbooks will have even **more** patterns in them because of the increased emphasis on algebraic reasoning.
- Teachers will be teaching pattern activities to children at all grade levels.
- Mathematics Their Way is a very popular source of activities in kindergarten and first grade. Other textbooks and sources of activities have copied such MTW activities as Calendar Math and Hundreds Day.
- The MTW activity book can be purchased through Addison-Wesley or even Amazon.com. However, the best way to learn about this program is to attend one-week workshops that are offered throughout the United States every summer.
- Children need experiences with a variety of patterns.

- Textbooks often help to create the misconception that rule for the pattern is the first number of the pattern.

- Algebraic Reasoning will be discussed in more detail in a later section. The concept of function will also be explored.

1.3 Mathematical Reasoning

In a study of fifth-grade students, Reid (2002) concluded that at this developmental level students may not be ready to engage in formal mathematical reasoning, specifically deductive reasoning.

The NCTM Standards (2000) encourages teachers to engage students in mathematical reasoning at all grade levels. Students should be encouraged to justify their mathematical thinking, not just give answers.

The problem solving done in this course provides a good source of instances where students may have engaged in mathematical reasoning. Problem #7 in Section 1.1 is a nice example where students have to justify their solution. Other nice examples are cases where there were different justifications presented for the same problem.

Teaching “key” words does not promote mathematical understanding as the example of the word problem in this section illustrates.

It sounds simple: **mathematics should be a sense-making activity.** However, in much of the current practice of teaching mathematics at the elementary level, children are not engaged in sense making.

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Chapter 2 Sets

2.1 Set Theory

A Historical Perspective

Sometimes the current reform efforts in mathematics education are referred to as 'New Math'. This is not the case! 'New Math' was based on the structure of mathematics. Current reform-oriented practices are based on **how children learn mathematics**.

What is number?

Put four objects on a table (e.g., four erasers, four pieces of chalk, four books, etc.) Ask, "What do you see?" When students mention they see four _____, ask, "Where is the four?"

The point is that the "four" is not <u>in</u> any of the objects. It is an idea we impose on the situation by putting the objects into a relationship, i.e., by mentally putting them into a group or <u>set</u>.

Assign for homework: **Define 7.**

In set theory, 7 is defined as the cardinality of all sets containing seven elements. Mathematicians do not follow the rules for defining words either! Number is difficult to define. Yet we expect 4 and 5 year olds to understand number.

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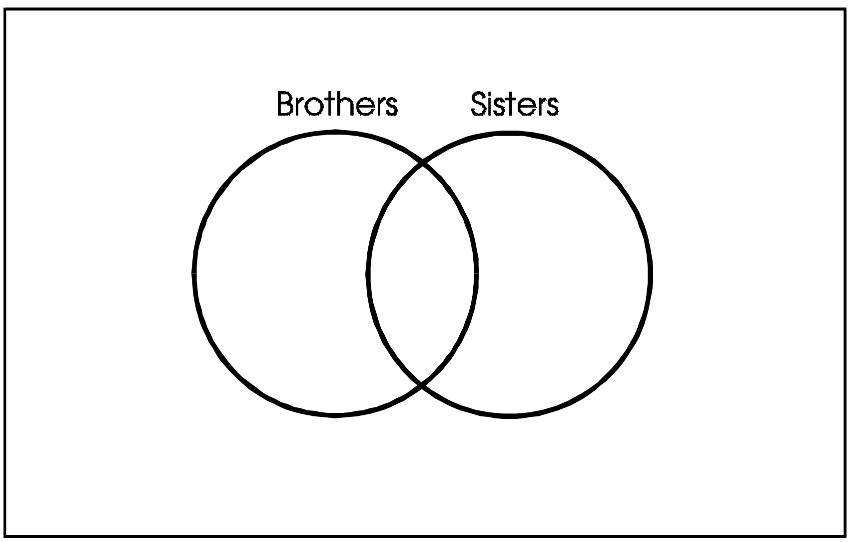
2.2 Venn Diagrams

Try the Venn diagram activity of brothers and sisters in your class. Use the total number in each set rather than individual names.

- Draw the Venn diagram on the board.
- Record the total number of students in the class as the Universal set.
- Ask and record how many have brothers only in the appropriate space.
- Ask and record how many have brother and sisters.
- Ask and record how many have sisters only.
- Ask the class to determine how many in this class do not have any brothers and sisters without asking the question directly?
- Check the solution by asking the class how many do not have any brothers or sisters.



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