Chapter 5 Integers

Where do you use or need negative numbers? List some situations where negative numbers are used.

____

____

____

____

____

Children in third grade have been known to "invent" negative numbers in the process of solving subtraction problems such as 83 – 27. These children typically reason as follows: "80 – 20 is 60. Then 3 – 7 goes 4 below 0 because 3 – 3 is 0 and 7 is 3 + 4 so there are still 4 more to take away. Then I do 60 – 4 and get my answer, 56."

From Silver Burdett Ginn, Grade 5 (New Jersey: Simon & Schuster, 1998) p. 471.

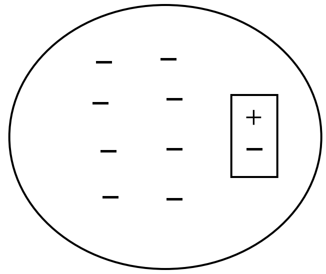
Although some children invent them on their own, negative numbers are not formally introduced until fifth or sixth grade. When presented with a "Find the Pattern" sequence like that below, most fourth graders will put 0 in the last blank.

11, __, 7, 5, __, 1, __

In this context, the possibility of a negative number does not yet occur to them. However, in different contexts, children may discuss temperatures below freezing or a loss of yardage on a football play meaningfully. As a teacher it is important to help children understand the meaning of negative numbers.

The main problems students experience with negative numbers center around the rules for performing calculations with them, specifically subtracting a negative number, multiplying or dividing a positive number and a negative number, and multiplying or dividing two negative numbers. In an effort to avoid simply stating the formal mathematical rules that apply to integer arithmetic, textbooks have taken to using various means of representation to help students make sense of these rules. One such model uses black and red chips to represent positives and negatives. Sometimes this is elaborated with the notion of credits and debits so that one black chip represents a \$1 credit and one red chip represents a \$1 debit.

This is quite different than the approach of allowing students to use manipulatives, diagrams, and well-chosen problem contexts as aids in constructing <u>their own</u> computational algorithms. In the latter case, we have noted that once children have developed the appropriate conceptual basis, the teacher may introduce the standard computational algorithms, confident that children's conceptual understanding will enable them to make sense of these algorithms. Using representations to model rules for computing with negative numbers is different because the representations are intended to lead children to a set of <u>predetermined</u> rules. Not only must children learn the conventions of operating in the chip model, but also they must figure out how the model maps onto, or corresponds to, rules for operating with positive and negative numbers. In other words, the representational model is intended to serve as an "intermediary" that will make the rules transparent. Unfortunately, such models can make the rules more opaque. First, these models seldom "hold up" throughout the full range of computational situations with positive and negative numbers. For example, try to figure out how the black and red chip (credit and debit) model can be used to illustrate $8 \div -2 = -4$. Second, children's thinking is sometimes constrained by the conventions of the model. In order to use the chip model to calculate $-8 - 1$, students must represent -8 as 9 red (negative) chips and 1 black (positive) chip (or 10 reds and 2 blacks, 11 reds and 3 blacks, etc.). Then, supposedly, one black chip may be taken away, leaving a set of chips that represents -9. However, we have found some students who, although able to conceive of -8 as $-9 + 1$, or 9 black chips and one red chip, seem unable to operate with this representation. For them, the red chip and the "extra" black chip are either "imaginary" or "don't really exist" or else they see these two chips as locked together in such a way that they cannot be uncoupled.



-8 as 9 negatives and one positive with an “imaginary” or “bonded” $+,-$ pair

For these students, the conventions of the model seem to get in the way of their efforts to learn the rule for subtracting a positive number from a negative number.

Those teaching arithmetical operations with positive and negative numbers are faced with a difficult choice: just present the rules as “the way it is” and something to be memorized, knowing that some students will forget or apply the rules incorrectly, or use a representational model such as the chip model in order to provide a context for the rules, knowing that some students will struggle with the conventions of the model. An alternative to either of these choices that is sometimes recommended involves using patterns to suggest the appropriate rules and encouraging students to draw on their previous knowledge of operations with whole numbers (e.g., the relationship between multiplication and division, the commutativity of multiplication) in order to extend the rules. For example:

$2 \times 4 = 8$	$4 \times -2 = -8$
$2 \times 3 = 6$	$3 \times -2 = -6$
$2 \times 2 = 4$	$2 \times -2 = -4$
$2 \times 1 = 2$	$1 \times -2 = -2$
$2 \times 0 = 0$	$0 \times -2 = 0$
$2 \times -1 =$ ___ (-2)	$-1 \times -2 =$ ___ (2)
$2 \times -2 =$ ___ (-4)	$-2 \times -2 =$ ___ (4)
$2 \times -3 =$ ___ (-6)	$-3 \times -2 =$ ___ (6)
$2 \times -4 =$ ___ (-8)	$-4 \times -2 =$ ___ (8)

Then for the problem $8 \div -2 =$ ___, one may reason as follows:

$8 \div -2 =$ ___ $\Leftrightarrow$ $-2 \times$ ___ $= 8$; since $-2 \times -4 = 8$, then $8 \div -2 = -4$.

While this pattern approach avoids the sometimes-artificial conventions of the representational models, it lacks the “real-life” context of the models that may help some students remember the rules. Further, some students may not “see” the patterns the teacher intends. Thus, like the chip model, the pattern approach is just one possible “trick” for revealing the rules for adding, subtracting, multiplying, and dividing when negative numbers are involved.

5.1 Addition and Subtraction of Integers

A number line is another tool that can be used to illustrate addition and subtraction of integers to students. While it works fairly well for adding and subtracting positive integers and even for adding negative integers, it runs into some difficulty when it comes to subtracting a negative integer.

From Everyday Mathematics SRB, Grade 5 (Chicago: Everyday Learning, 2002) p. 92.

How could you explain 4 − (−2) to children? A double negative in the English language provides a nice example. **If I am not, not going to the store, I am going to the store!**

As an aside, why should you use parentheses to separate two signs next to each other?

If we write 4−−2, some may see one big minus sign and interpret this as 4—2 instead of 4 − (−2) as intended. Similarly, for 5 + −3 it is better to write 5 + (−3).

Thus, we use parentheses for neatness and to avoid confusion.

Another possible source of confusion involving negative numbers arises when absolute value symbols are used. For example, −½−7½

Students may argue that two negatives make a positive, so the answer is 7.

Absolute value symbols may be considered to act like parentheses in regards to the order of operations, in that we perform any operations inside of the symbols first, then we take the absolute value of the result. For example, ½7 - 4½is the same thing as ½(7 - 4)½= ½3½= 3.

5.2 Multiplication and Division of Integers

It is important to explain the rules for the multiplication and division of integers even though these rules are very basic. Children need to know why mathematics works or at least see an explanation of why it works. If students are familiar with thinking of 2 × 3 as indicating the addition of 2 groups of 3 (i.e., 3 + 3), then multiplication of a whole number times a negative number can be explained as repeated addition.

$$3 \times (-4) = -4 + (-4) + (-4)$$

However, a negative number times a whole number (e.g., −2 x 4) is much more difficult to explain as repeated addition. What does –2 x 4 mean? –2 groups of 4? Since multiplication is commutative, the numbers can be switched and repeated addition can again be used to explain the process.

Another way to illustrate a negative times a negative is to employ the "continue the pattern" approach discussed above.

For example:

−5 x 2 = −10
−5 x 1 = −5
−5 x 0 = 0
−5 x –1 = __
−5 x –2 = __

For division the same principles and rules apply.

Oftentimes we give children problems that are intended to illustrate multiplication and division of negative numbers, but children do not think of the problems in these terms.

If I lost 4 pounds a week for 3 weeks, how many pounds did I lose?

Mathematically, we might think of the problem as 3 x −4 = −12, but how do you think children and most adults really solve the problem? Probably, they think 3 x 4 = 12 and then make the solution a 12-pound loss.

If a school lost 10 students a year, how many more students did the school have 2 years ago?

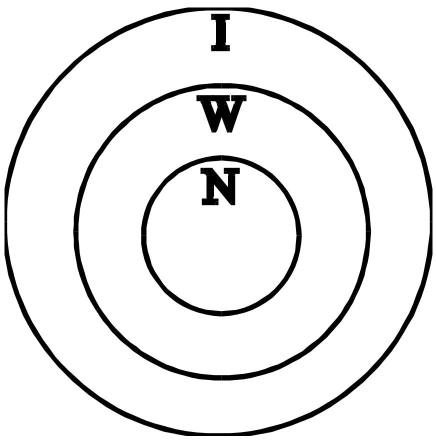
Mathematically, −2 x –10 = 20, but especially here students will just multiply and think of it as 2 x 10 = 20.

The way children think about problems involving negative numbers may be very different than the intended formal approach to mathematical thinking.

5.3 Properties of Integers

The integers are closed for subtraction (the difference between any two integers is an integer), but they are not closed for division (e.g., −2 ÷ 4 is not an integer).

One of the beauties of mathematics is that once we have properties for number systems and we add more number systems, the same properties apply. A Venn diagram of Natural Numbers, Whole Numbers, and Integers can be used to illustrate the relationship among the number systems.



For example, the commutative property works for Natural Numbers and it will also be true for Whole Numbers and Integers. As illustrated in the drawing, the Natural Numbers are a subset of the Whole Numbers. Therefore, the properties of Natural Numbers remain. We only gain properties—we never lose them.

A property we gain with integers is the additive inverse. While we may not call it a property with children, a typical question is to ask children for the additive inverse of various numbers:

7 19 −5 −102 0

Chapter 5
Questions for Discussion and Review

5.1

1. In what contexts do you think children have an understanding of negative numbers?
2. How would you introduce addition and subtraction of negative numbers to children?
3. What does the expression "go into the red" mean? How does it relate to integers?

5.2

1. How might children solve or think about the following problem?

   Jim's football team lost 5 yards on 2 consecutive plays. How many yards did the team lose?

2. How would you explain this pattern to children?

   $2(-4) = -8$
   $1(-4) = -4$
   $0(-4) = 0$
   $-1(-4) = \_\_$
   $-2(-4) = \_\_$

5.3

1. What is the additive inverse of a number?
2. As we build our number systems from Natural Numbers to Integers, what happens to the properties associated with each system?

Chapter 6 Rational Numbers – Fractions

6.1 Fractions

Overview

What is the difference between a fraction and a rational number?

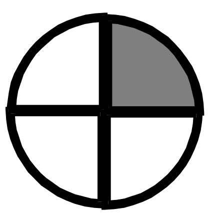
Fractions are a difficult topic for children and many adults! Even calculus teachers complain about their students not understanding fractions when problems involve fractional expressions.

Research shows that children develop an understanding of 1/2, 1/3, and 1/4 in that order but their understanding of other fractions drops off rapidly, especially when numbers other than 1 are used in the numerator.

Research also indicates that children will develop a deeper conceptual understanding of fractions by using multiple modes of representation—pictorial, manipulative, verbal, real-world, and symbolic. Emphasis should also be placed on the use of multiple physical models and the connection between them. For example, children may be given a fraction model with 2/3 of a circle shaded and then asked to show that same fraction with a set of chips (Cramer, Post, & delMas, 2002).

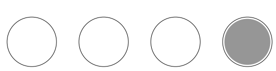
What models do you remember using in learning about fractions? A popular model has always been a pizza or a pie.

What does this picture represent?



This is the most common representation of a fraction and it is a fraction of a **whole** or of a **continuous region**.

What does this picture represent?



This is an example of a fraction of a **set** or of a discrete set of objects

What is the difference between the two pictures?

Most children have many experiences with fractions of a whole but they also need more experiences with fractions of a set! One is not more difficult, it is just that in most cases children do not get as much experience with fractions of a set.

From Silver Burdett Ginn, Grade 3 (New Jersey: Simon & Schuster, 1998) p. 408.

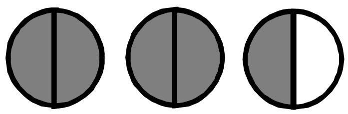
Your textbook may discuss fractions as locations on a number line. While this at times is a useful representation, it is not how children initially make sense of fractions.

Another conceptualization of fractions is as division. For example, the fraction 1/2 is the same thing as 1 ÷ 2. This is useful later on in mathematics but it probably does not make sense to children when they are first learning about fractions.

Improper fractions: This is not a good name for this type of fraction because there is nothing improper or wrong with an improper fraction.

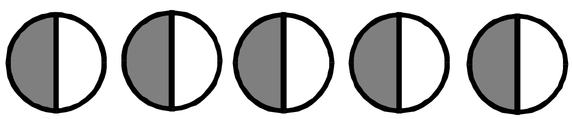
Make a drawing, using circles, to represent the improper fraction 5/2.

One solution is to draw 3 circles and shade in 5 half-circles.

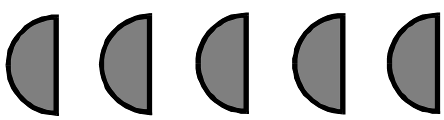


Some children might interpret this picture as showing 5/6 because 5 out of 6 parts are shaded. A confusing aspect of fractions is keeping track of the referent or whole that the fraction refers to. For the above drawing to represent 5/2, the referent is one whole circle as it was in the previous representation of proper fractions. Students are correct to say 5/6 if the whole is taken to be the 3 circles combined. A very confusing aspect of fractions!

Another representation may be 5 full circles cut in half with half of each shaded.



Still another representation is 5 half-circles.



<u>Fraction Concepts</u>

Children need to have a part/whole concept of natural numbers (cf. Chapter 3) before they can begin constructing fraction concepts. Once they are able to think of natural numbers as units composed of smaller units, students' informal notions of partitioning, sharing, and measuring may provide a starting point for developing the concept of fractions. In fact, sharing may play the role for fractions that counting does for natural numbers. Initially, many children are more concerned that each person gets an equal number of things than with the size of each thing. But early on in elementary school they become more aware of the size of the parts and can partition quantities into equal shares corresponding to halves, fourths, and eighths (National Research Council, 2001). Research indicates that children have sound informal knowledge of one-half and powerful strategies for halving. Studies also suggest that children in second grade are capable of formulating the concept of fraction, provided they are encouraged to use manipulatives and fractions are described orally by fraction words. Numerical symbols should not be introduced at this early stage of fraction concepts.

Experiences in partitioning are of critical importance in helping children construct an understanding of fractional numbers. Segmenting a whole into equal-sized parts, recombining the parts to form a whole, and iterating a part so as to measure with it are essential mental operations in the development of children's fractional meanings. Initial partitioning activities may use continuous regions such as circular pizzas or rectangular cakes. Thereafter, problems using discrete sets of objects may be presented (e.g., If these 6 marbles are one-fourth of all my marbles, how many marbles do I have?) Activities that enable children to use their prior knowledge of natural numbers to generate initial fraction conceptualizations are especially useful. For example, children's thinking about fractions can build from their multiplicative operations.

Children need to conceptualize fractions as quantities before they are introduced to conventional symbolic algorithms for adding, subtracting, multiplying, and dividing fractions. Problems posed in the context of money can help children view fractions as quantities. Children can focus their attention on the value of a bill or coin that is part of a larger bill or coin that represents the whole. Some sample questions of this type are:

1) A nickel is what part of a quarter?
2) A nickel is what part of a dollar?
3) A dime is what part of a dollar?
4) A dime is what part of two dollars?
5) A dime is what part of five dollars?
6) This nickel is one tenth of what I have under here [indicating some money covered under a cloth]. What amount of money is under the cloth?
7) These two nickels are two twentieths of what I have under here [indicating some money covered under a cloth]. What amount of money is under the cloth?
8) Which is bigger, one-tenth or one-twentieth of 1000 dollars?
9) Which is bigger, five-tenths or one-half of 1000 dollars? (Sáenz-Ludlow, 1994).

Questions such as the latter two allow children to use natural-number comparisons to generate fractional comparisons. For example, by determining that one-tenth of 1000 dollars is 100 dollars and one-twentieth of 1000 dollars is 50 dollars, a child may reason that one-tenth is larger than one-twentieth. In this way such questions enable children to take advantage of their natural-number knowledge to learn about fractions while at the same time avoiding the overgeneralization of natural-number properties that, in the absence of any meaningful context, often leads children to say one-twentieth is larger than one-tenth because twenty is larger than ten. Children's concepts of fractions take a good deal of time to develop. As noted above, children need to have a quantitative understanding of fractions before they can make sense of symbolic fraction notation and algorithms for performing operations on fractions. As part of this quantitative understanding, students should know something about the relative size of different fractions and they should know equivalents of one-half and other common fractions. Furthermore, children's informal written representations of fractions should be considered as transitional notations before the conventional symbolic notations are introduced.

What do children understand about fractions and how will you find out what they know? One means might be to interview a child asking varied questions about fractions.

Homework: Complete the following fraction interview.

Fraction Interview

Your assignment is to conduct an interview with a student and then write a description summarizing your assessment of that student's understanding of fractions. Try to choose a student who has some understanding of fractions but not a student with an advanced understanding. (Most students in grades 4-9 would fall into this category.) A high school student without a thorough understanding of fractions would be fine. Try to figure out what they know about fractions. Use specific examples to justify your conjectures about their understanding.

Suggested Questions:

1. What fractions do you know or use?

2. What does 5/8 mean?

3. What does 5/3 mean?

4. Which is larger, 1/8 or 1/3?

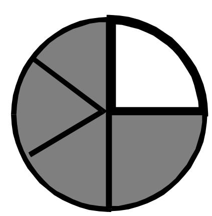
5. Which is larger, 2/3 or 3/4? How do you know?

6. What is the largest fraction you know? What is the smallest fraction you know?

7. Draw a picture representing 3/4.

8. Draw a picture representing 5/2.

9. Does this picture represent 4/5? Why or why not?



10. What is 1/2 + 1/3?

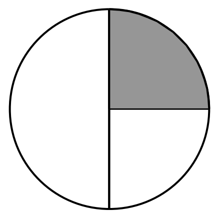
Difficulties Students Encounter With Fractions

The National Assessment of Educational Progress (NAEP) documents a low level of performance on fraction computation tasks and a lack of understanding of fractions among nine-, thirteen-, and seventeen-year-olds. Asked to estimate $\frac{12}{13} + \frac{7}{8}$, 55% of thirteen-year-olds chose either 19 or 21 from among the answer choices 1, 2, 19, and 21. Similarly, in a videotaped interview, a ninth-grade general math student estimated 3/4 + 3/4 was less than 1. These answers stem from a common error students make when adding fractions: adding numerators and denominators (e.g., $\frac{1}{3} + \frac{2}{5} = \frac{3}{8}$). This may result from lack of a conception of the implicit whole to which the fractions are connected and from over generalizing properties of operations with natural numbers. In fact, many properties of fractions and rules for performing operations on fractions conflict with well-established ideas about natural numbers. For example, students may believe that $\frac{1}{3} > \frac{1}{2}$ because 3 > 2. Or they may think that $\frac{4}{5}$ and $\frac{5}{6}$ are the same because, in each case, the numerator is one less than the denominator. In other words, students sometimes use properties learned from working with natural numbers even though those properties do not apply to fractions (National Research Council, 2001). However, we noted in the previous section that it is possible to design activities that make it possible for children's natural-number knowledge to help rather than hinder their learning of fraction concepts.

The algorithm for adding fractions is not the only procedure for operating on fractions that students do not understand. Significant portions of students do not understand most of the computational procedures they use with fractions. For example, students may know the rule for converting a mixed number to an improper fraction (e.g., $3\frac{1}{4} = \frac{(3 \cdot 4) + 1}{4} = \frac{13}{4}$), but they often do not understand why they do this. Likewise, students do not understand why they "invert and multiply" to divide fractions. As a result of this lack of understanding, students' algorithms develop "bugs," such as inverting the dividend instead of the divisor before multiplying or multiplying fractions by "cross multiplying." **These errors typically result when students are forced to try to memorize the steps of an algorithm that does not make sense to them**.

Part of the reason for the aforementioned difficulties is that the use of manipulatives in developing the concept of fraction is abandoned too quickly and an insufficient amount of time is spent on the concept of fraction and on ordering and equivalence of fractions before operations on fractions are introduced. Thus, students who don't understand what fractions are asked to do things to them. This results in the meaningless application of rote procedures and in the inability to assess the reasonableness of the results of those procedures (Bezuk, 1988). Furthermore, for a conventional arithmetical algorithm to become meaningful to a child, it must represent the coordination of the child's thinking and conventional notation. Introducing conventional notations too soon may impede children's learning by having them use symbols that are foreign to them to represent their thinking (Sáenz-Ludlow, 1995).

Another source of students' difficulties in developing sound fraction concepts stems from the way textbooks typically introduce fractions. Textbooks depict fractions using pre-partitioned shapes and then ask students to identify various fractions or show them by shading (again using pre-partitioned shapes), but research shows students complete these exercises without focusing on the geometrical properties of the whole or the parts. Textbook exercises do not provide opportunities for students to use physical models or to draw their own diagrams to solve problems and thus they do not engage students in activities where they must do the partitioning. Consequently, some students don't realize that the pieces of the whole have to be the same size. For example, some students will say that one-third of the figure below is shaded.



This type of response suggests that when students only encounter figures that are already partitioned into equal-sized pieces, they can obtain correct answers without constructing the fundamental notion that a fraction such as $\frac{1}{3}$ refers to one out of three equal-sized pieces, not just one out of three pieces. Thus, this type of response suggests that students need more fair-sharing partitioning experiences.

In general, textbooks expect students to abstract fraction concepts before they fully understand them. For example, students are told that one of two equal-sized parts is called "one-half" and written "$\frac{1}{2}$." The problem is that this symbol is confusing for students who have previously dealt only with natural numbers because it requires them to recognize that the numbers 1 and 2 represent how many of how many equal-sized parts, but, at the same time, these numbers represent another number, that is, the number $\frac{1}{2}$ (Bezuk, 1988). Further, students must learn that the two numbers that make up a common fraction (numerator and denominator) are related through multiplication and division, not addition.

Because fractions involve complex part/whole relationships and some situations involve not only multiple parts, but also multiple wholes, children sometimes "lose track of" the whole. For example, when asked to share two whole pizzas among 4 people, a child may cut each pizza in half, distribute the four halves to the four people, and conclude that each person gets $\frac{1}{4}$. In other words, the child sees each person as receiving one out of four equal parts, but loses sight of the fact that each part is $\frac{1}{2}$ of a pizza (Charles & Nason, 2000). Additionally, when presented with the problem of sharing 3 pizzas among 4 people, students often give the following answers for the amount of pizza each person gets: "$\frac{1}{4}$ of a pizza," "$\frac{1}{4}$ of each pizza," "$\frac{3}{4}$ of three pizzas," or "$\frac{1}{4}$ of all the pizzas."

For the problem of sharing 4 cookies among 3 people, they often give the answer of "$1\frac{1}{3}$ of two cookies." (Which of these answers would you consider incorrect? Which, if any, would you consider incomplete, but not necessarily incorrect?) These responses indicate that students have difficulty understanding the unit (or whole) to which a fraction is referring.

The variety of uses fractions have is certainly one of their great benefits, but this also contributes to students' difficulties in learning about them. Students must learn to relate and distinguish between multiple interpretations of a fraction. For example, $\frac{3}{4}$ may indicate

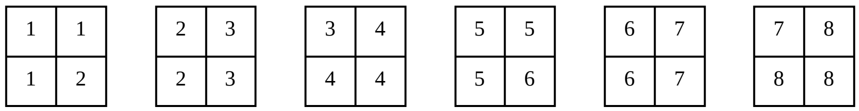
> a part/whole relationship (3 out of 4 equal-sized parts)
> a quotient ($3 \div 4$)
> a measure ($\frac{3}{4}$ of the way from the beginning of the unit to the end of the unit)
> a ratio (3 cats for every 4 dogs)
> an operator that may be applied to another number ($\frac{3}{4}$ of 8) (National Research Council, 2001).

Understanding the connection between fractions and division seems to be especially difficult for students. That is, they do not understand the connection between $\frac{3}{4}$ as a number and $3 \div 4$. (How are these connected?) The ratio interpretation can also present significant difficulties. For example, in the whole group of cats and dogs for which there are 3 cats for every 4 dogs, the fraction of cats relative to the whole is $\frac{3}{7}$ and the fraction of dogs relative to the whole is $\frac{4}{7}$. In this case, $\frac{3}{4}$ is actually comparing two <u>parts</u>, not a part and the whole.

The final difficulty we mention here is perhaps not so much a difficulty as a phenomenon for teachers to be aware of. Many teachers have observed that students can solve what the teacher would call multiplication and division fraction word problems, but the students don't see the problems as involving multiplication or division (i.e., they don't write multiplication or division number sentences for the problems). This may be related to their natural-number notions of multiplication and division, namely that multiplication makes bigger and division makes smaller. In the case of division, it may also stem from thinking only in terms of the sharing model of division. For example, children may believe that you can't divide 3 by 4 because the dividend must be greater than the divisor or they may believe that the number sentence $4 \div \frac{1}{4}$ does not make sense because the divisor must be a whole number (how can you share 4 things among $\frac{1}{4}$ people?) or they may believe that $1 \div \frac{1}{2} = 2$ is impossible because "division makes smaller," that is, the quotient must be less than the dividend. In spite of such beliefs, children are able to construct viable strategies for solving problems like the following:

> You are giving a party for your birthday. From Ben and Jerry's Ice Cream Factory, you order 6 pints of each variety of ice cream that they make. If you serve _ of a pint of ice cream to each guest, how many guests can be served from each variety?

A typical solution produced by sixth graders was to draw a picture to represent six pints of ice cream, separate each pint into four equal sections, and distribute three of those sections at a time to guests. From this they concluded that eight guests could be served from each variety.



Although, on the surface, the students' solution strategies did not appear to vary greatly, the number sentences they wrote for this problem did:

> $24 \div 3 = 8$ (because there are 24 pieces, 3 pieces to a serving, so 8 people can be served)

$8 \times \frac{3}{4} = 6$ (because 8 servings of $\frac{3}{4}$ of a pint gives you 6 whole pints)

$\frac{3}{4}+\frac{3}{4}+\frac{3}{4}+\frac{3}{4}+\frac{3}{4}+\frac{3}{4}+\frac{3}{4}+\frac{3}{4}=6$ (because $\frac{3}{4}$ each gives you 6 whole pints)

$6-\frac{3}{4}-\frac{3}{4}-\frac{3}{4}-\frac{3}{4}-\frac{3}{4}-\frac{3}{4}-\frac{3}{4}-\frac{3}{4}=0$ (because you take away $\frac{3}{4}$ of a pint for each serving and you can do this 8 times)

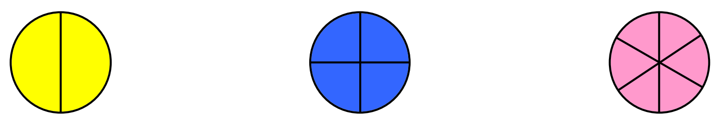
A teacher who was expecting students to write the number sentence $6 \div \frac{3}{4} = 8$ for this problem may feel that the students' number sentences do not match what is going on in the problem. However, the students' number sentences do match what was going on in the problem <u>for them</u>. By writing number sentences involving division of whole numbers, multiplication of a fraction by a whole number, and repeated addition and subtraction of fractions, the students were expressing the variety in their levels of understanding and ways of thinking about multiplication, division, and fractions. This multiplicity of ideas provided the teacher an opportunity to help children see how these ways of thinking about the problem are related to each other and to division of fractions (Schifter, 1997). (Think about how multiplication, subtraction, and addition are all involved in the standard algorithm for dividing <u>whole</u> numbers.) If the students are familiar with the measurement concept of division, a teacher in the above situation might, after encouraging students to compare the different number sentences, say, "Here's another number sentence we could write for this problem: $6 \div \frac{3}{4} = 8$, because there are 8 three-fourths in 6." Note that this is an example of why it is important for students to have experiences with both the sharing concept of division and the measurement concept of division. If students have only thought in terms of the sharing concept, the teacher's justification of why the number sentence $6 \div \frac{3}{4} = 8$ is appropriate for this problem is not likely to make sense to them.

<u>Fraction Concept Activities</u>

As the discussion of students' difficulties with fractions suggests, students need more experiences

a) partitioning a unit into equal-sized parts
b) comparing fractions (i.e., determining the relative size of two or more fractions)
c) generating equivalent fractions
d) solving fraction problems in "real-world" contexts

Several activities aimed at helping students develop an initial understanding of fractions were discussed in the section **Fraction Concepts**. Another activity that can help children focus on the importance of the unit (whole) in relation to the parts involves using fraction circles to pose questions like the following:



If the cost of one yellow piece is 1 dollar, what is the cost of one blue piece?
If the cost of one blue piece is 1 dollar, what is the cost of one pink piece?

When students produce different partitions of the same shape, questions like that below aid in their development of logical part/part and part/whole relationships:

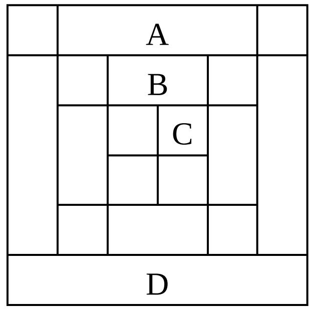
cookie A cookie B

If one child gets one piece of cookie A and another child gets one piece of cookie B, will they get the same amount? Who will get more? Why? (Pothier & Sawada, 1990).

To help children construct equivalent fractions, the teacher can present problems that require the comparison of two fractional units of the same whole. For example:

If Grandpa gave one-fifth of his money to Sam and one-tenth of his money to Sue, what part of his money did he give to the two children?

The simultaneity of different partitions of the same whole (fifths and tenths) fosters the need to correlate them and leads to the generation of equivalent fractions. Tasks based on structured patterns like that below can also facilitate the generation of equivalent fractions.



D is what part of the large square?
B is what part of the large square?
C is what part of the large square?

By sequencing the questions appropriately, children are required to generate new partitions "on top of" previous ones. This leads to the construction of a nested system of partitions that allows children to find equivalent fractions (Sáenz-Ludlow, 1994).

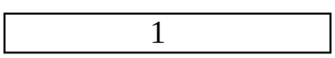
A hands-on activity to help children develop a conceptual understanding of fractions is Fraction Bars.

Small-Group Activity: **Fraction Bars**

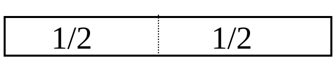
Suggestions:

Cut up strips of equal length, approximately 8 per student. (The paper slicer works nicely. Try to make each strip approximately 1 inch wide.)

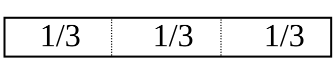
Pass out the 8 strips to **each** student. Have students label one of their strips 1.



Fold the next strip in half and label each section 1/2. Suggestion: put a line or dotted line through the fold to highlight it. If a strip is misfolded, start over with another strip. The folds are very important.



Fold the next strip into thirds. This can be done by folding the strip like a letter to be put in an envelope or by making <u>1 and 1/2</u> loops with the strip and folding it at the ends. Label each section 1/3.



Next fold a strip into fourths. Fold the strip in half once and then repeat the procedure. Label each section 1/4.

1/4	1/4	1/4	1/4

Fifths may be the most difficult for students to make, but for intuitive understanding fifths are essential for the subsequent activities.

Student can make fifths by making a circle with 2 and 1/2 loops. Crease or fold the strip wherever the ends of the strip fall. This is the most challenging strip!

1/5	1/5	1/5	1/5	1/5

Next make sixths. This can be done by first making thirds and then folding the thirds in half. Another way is to make 3 loops and fold the strip at the ends.

1/6	1/6	1/6	1/6	1/6	1/6

Sevenths are optional. They can be made by making 3 and 1/2 loops and creasing the circle at the ends of the strip.

The last fraction bar is eighths. Fold the strip in half three times.

1/8	1/8	1/8	1/8	1/8	1/8	1/8	1/8

Notice that sixteenths are not too difficult to make but other fraction bars may prove tricky.

Problem: **With a partner, using your fraction bars, find as many different ways as possible to make 1 or a whole**. Pretend that you do not know how to add fractions by finding a common denominator!

As an example, 1/2 and 2/4 (or children may think of 2/4 as two 1/4s)can be put together to make a whole. Record your solutions. For the previous solution, write:

1/2 + 2/4 = 1
or
1/2 + 1/4 + 1/4 = 1

As you are working make sure that you use three fraction bars to make a whole and also that you use the fifths fraction bar.

Homework: **Using your fraction bars, make inequalities. Do not use only 1's in the numerator. For example: 2/3 < 3/4.**

Cuisenaire rods are another manipulative that is sometimes used to teach fractions. You should be aware of them. They are not marked off in length so children have to solve the problem of determining the relationship among them.

From Math Trailblazers, Grade 4 (Dubuque: Kendall/Hunt, 1997) p.327

6.2 Addition and Subtraction of Fractions

Common Difficulties

Why are operations with fractions so difficult for children? Research shows that an expert or experienced person in mathematics might look at the problem 2/3 + 4/5 as one operation on two numbers. An inexperienced child might view this as 4 numbers with 3 operations. (Operations may not necessarily mean interpreting the fraction bar as the division operation, but as something that must be done.) Most children in elementary school are not ready to work with 4 numbers and 3 operations.

Another common difficulty with fractions is illustrated by the following problem: **What is _ of a large pizza and _ of a small pizza?** Is the solution 1 whole pizza? These fractions cannot be added because they do not refer to the same whole. Operations on fractions can be very confusing.

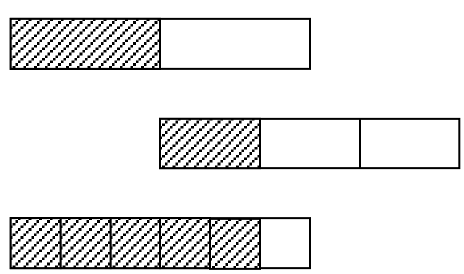
What do you get when you add _ of an apple and _ of an orange?

Models for Adding and Subtracting Fractions

The addition or subtraction of fractions with common denominators (e.g., 2/8 + 3/8 = 5/8) can be demonstrated easily with pictures and diagrams. It is somewhat more challenging to demonstrate the addition or subtraction of fractions without common denominators (e.g., 1/3 + 1/2 = 5/6).

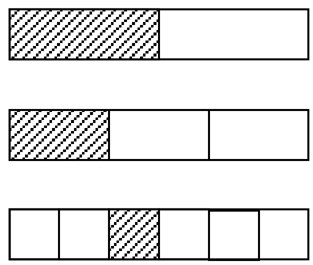
It is important to provide a model for adding fractions with unlike denominators to help children develop an understanding of the process. Fraction bars offer one means of doing this.

To add 1/3 + 1/2, take the fraction bar divided into thirds and fold it so you have a strip 1/3 long. Do the same with the bar divided into halves to make 1/2. Put the two strips together. Holding them together compare them to the other fraction bars until the 1/3 + 1/2 strip lines up exactly on a fold. If you hold up eighths, the end falls in between folds, but if you compare it to sixths, it falls at the last fold, which is 5/6.



1/2 + 1/3 = 5/6

You can illustrate subtraction in the same way. For 1/2 − 1/3, hold up the 1/2 strip and put the 1/3 strip over it. Now you are trying to find the difference of the two strips, which should match up with 1/6.



1/2 − 1/3 = 1/6

Pick examples carefully—if you try 1/2 − 1/5 there is no fraction bar (tenths) in the set that will work.

Activities for Addition and Subtraction of Fractions

When beginning addition of fractions, use word problems such as the following:

> Tyra ate $\frac{1}{4}$ of a small cake and Michael ate $\frac{1}{3}$ of the cake. What part of the cake did the two of them eat altogether?

If children are encouraged to use manipulatives, such as fraction circles or fraction bars, or to make diagrams to solve problems like this, they can draw on their ideas about partitioning and equivalent fractions—developed from the types of activities discussed in the previous section—and construct for themselves the need to obtain a common denominator in order to add two fractions.

When adding fractions, it is useful to ask students to estimate the size of the sum before solving the problem with manipulatives or diagrams. This estimate not only engages their understanding of the meaning of the fractions involved, but it will also help them judge the reasonableness of the answer they obtain. For instance, in the problem above, a student may reason that the sum should be greater than $\frac{1}{2}$ because $\frac{1}{4}+\frac{1}{4}=\frac{1}{2}$ and $\frac{1}{3}$ is larger than $\frac{1}{4}$. Thus, an answer of $\frac{2}{7}$, obtained by adding numerators and denominators, is not reasonable here.

6.3 Multiplication and Division of Fractions

<u>Common Difficulties</u>

What else is different about fractions?

In third and fourth grade students learn intuitively that multiplication makes numbers bigger. Teachers often say this to their students and even if they do not, most students intuit the idea. It is true for whole numbers.

6 x 2 = 12, which is larger than 6.

But this same idea does not hold for fractions. 6 x 1/2 = 3, which is smaller than 6.

Likewise, students learn intuitively that division makes numbers smaller.

12 ÷ 2 = 6, which is smaller than 12.

But again, this idea does not hold for fractions. 12 ÷ 1/2 = 24, which is larger than 12.

From Silver Burdett Ginn, Grade 5 (New Jersey: Simon & Schuster, 1998) p. 408.

Models for Multiplying and Dividing Fractions

It is important to try to explain multiplication of fractions. Multiplication of a whole number times a fraction can be explained as repeated addition.

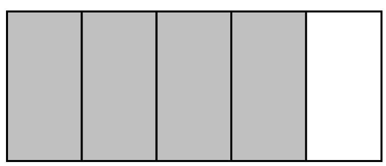
4 x 1/2 = 1/2 + 1/2 + 1/2 + 1/2

Since all the properties are retained (cf. Integers 5.3), multiplication of a fraction times a whole number can be reversed because of the commutative property.

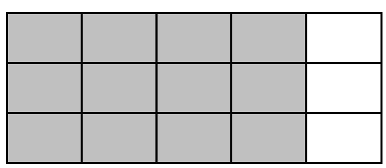
However, it is much more difficult to explain a fraction times a fraction. The repeated addition representation does not work very effectively when we look at 2/3 x 4/5. How do you show 2/3 of 4/5? If 3 x 4 is 3 – 4 times, then 2/3 x 4/5 is 2/3 – 4/5 times. Or if 3 x 4 is 3 groups of 4, then 2/3 x 4/5 is 2/3 groups of 4/5. Do these make sense?

One way to demonstrate multiplication of fractions between 0 and 1 is with the following diagram. This is also a good model for multiplication of decimals when the decimals are in tenths.

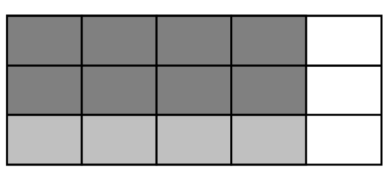
To solve 2/3 x 4/5 (2/3 of 4/5) using a diagram, begin by drawing a rectangle, partitioning it vertically into fifths, and shading in four-fifths.



Now partition the rectangle horizontally into thirds.



This partitions the 4/5 (the shaded region) into thirds so we can now shade in 2/3 of the 4/5.

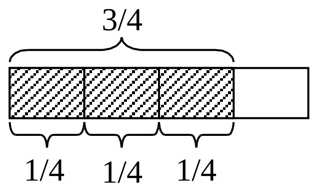


Thus, we see that 2/3 of 4/5 is 8/15.

Note that we began with 5 vertical strips. We divided each of the 5 vertical strips into 3 parts, giving us 5 x 3 (the product of the denominators) = 15 total pieces. Then, to find 2/3 of 4/5, we only wanted to count 2 of the pieces in each of the 4 shaded vertical strips. This gave us 4 x 2 (the product of the numerators) = 8 double-shaded pieces.

Division is much more difficult to explain.

If we have common denominators, then, for example, 3/4 ÷ 1/4 can be thought of as "how many 1/4's are in 3/4?". A picture may be useful to illustrate.



There are three 1/4's in 3/4 so 3/4 ÷ 1/4 = 3.

However, when the fractions do not have common denominators, it is more difficult to explain unless the fractions are changed to the common denominators.

Changing division of fractions to a complex fraction is one way to explain why we invert and multiply.

4/5 ÷ 2/3 =

$$\frac{4/5}{2/3} = \frac{4/5 \text{ x } 3/2}{2/3 \text{ x } 3/2} = 4/5 \text{ x } 2/3$$

There is also a way to illustrate division of fractions with paper folding. For 1/2 ÷ 1/4, fold a paper in half and then fold the half in half again. Upon unfolding, one may see that there are two 1/4's in 1/2. However, this model only works well if the answer is a whole number.

Fractions with 0

0/5 = ? 5/0 = ? 0/0 = ?

Preservice teachers should be able to explain each of these. One way is to first change each of these fractions to division and then change each division to multiplication as done in Chapter 3. A major difference is that 5/0 is undefined because there are no solutions but 0/0 is undefined because there are an infinite number of solutions.

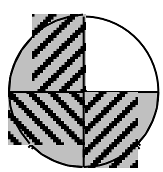
0/5 = ?	5/0 = ?	0/0 = ?
0 ÷ 5 = ?	5 ÷ 0 = ?	0 ÷ 0 = ?
5 x ? = 0	0 x ? = 5	0 x ? = 0

<u>Activities for Multiplication and Division of Fractions</u>

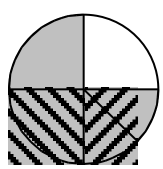
For multiplication of fractions, begin with word problems that ask students to find fractional parts of whole numbers, and then move to problems that ask them to find part of a part of a whole. For example:

> You have $\frac{3}{4}$ of a whole cake. You and your friends eat $\frac{1}{2}$ of that amount. What part of the whole cake did you and your friends eat?

Three actual student solutions to this problem are described below:



"Three-fourths is the same as $\frac{1}{4}+\frac{1}{4}+\frac{1}{4}$. So I took half of $\frac{1}{4}$, which is $\frac{1}{8}$, and three times that is $\frac{3}{8}$."



"Three-fourths is $\frac{1}{2}+\frac{1}{4}$. So I took half of $\frac{1}{2}$, which is $\frac{1}{4}$, and half of $\frac{1}{4}$, which is $\frac{1}{8}$. Since $\frac{1}{4}$ is the same as $\frac{2}{8}$, I added $\frac{2}{8}$ and $\frac{1}{8}$ and got $\frac{3}{8}$."

"I changed $\frac{3}{4}$ to $\frac{6}{8}$ and took half of $\frac{6}{8}$, which is $\frac{3}{8}$." (Warrington & Kamii, 1998).

Note that the second pictorial solution here involves the coordination of multiple partitions of the same whole, that is, the previously discussed idea of (mentally) placing one partition on top of another one. In this way the student was able to generate an equivalent fraction for $\frac{1}{4}$ that provided a common denominator and made it possible to add the two parts.

From Math Trailblazers, Grade 5 (Dubuque: Kendall/Hunt, 1997) pp.290,291.

6.4 Properties of Rational Numbers

A new property is multiplicative inverse, also referred to as the reciprocal.

Can you list the natural numbers in order from least to greatest? Whole numbers? Integers?

Try to list the rational numbers in order from least to greatest. For example, we might start as follows:

1/2, 1/3, 1/4, 1/5, 1/6, 1/7, 1/8, 1/9, …

But between 1/7 and 1/8 there is another fraction, in fact more than one fraction. We can show this by finding a common denominator, 1/7 = 8/56 and 1/8 = 7/56, and then multiplying by 2/2 to get 14/102 and 16/102. Now we see that 15/102 lies between 1/8 and 1/7. Another way to show this is to begin by multiplying by 2/2, which gives 2/14 and 2/16. This means that 2/15 must lie between 1/8 and 1/7. We can keep doing this forever (i.e., now we could find a fraction between 2/14 and 2/15). Therefore, between any two fractions, there is always another fraction. This is called the Denseness Property of Rational Numbers. This idea is used in calculus and higher mathematics analysis.

With children we simply ask them to find a fraction between two other fractions and do not talk about denseness.

For each point on the number line is there a corresponding rational number? For each rational number there is a point. If we enlarge the number line, say a million times, and look at the points that make up the line, we would see many points that do not correspond to rational numbers. For instance, we do not have a rational number for the points $\sqrt{2}$,$\sqrt{3}$, π, etc. We need **real numbers** to obtain a 1-1 correspondence with the points on the number line.

Returning to closure, the property of closure along with division drives our creation of rational numbers. Rational numbers (excluding 0) are closed with respect to division.

Chapter 6
Questions for Discussion and Review

6.1

1. What are two conceptualizations of fractions?
2. What are some pictorial and physical models of fractions?
3. What do first and second grade students learn about fractions?
4. What did you learn about children's understanding of fractions by doing the fraction interview?
5. Why would you have children make fraction bars before you had them use manufactured fraction bars?
6. Draw a picture representing 12/5.

6.2

1. How can you use fraction bars to show addition and subtraction of fractions with unlike denominators?
2. What are some addition and subtraction problems involving unlike denominators that children could solve with the fraction bars you have made?
3. Why can't you add 1/2 of a large pizza and 1/2 of a small pizza?
4. Explain how a child sees 2/3 x 4/5 as three operations with 4 numbers.

6.3

1. Is it okay to teach third-grade students that multiplication makes bigger and division makes smaller?
2. Use a diagram to solve 3/4 x 5/6.
3. Why do you invert and multiply when you divide fractions?
4. Why is 0/0 undefined?

6.4

1. Why can't we list all the rational numbers in order from least to greatest?
2. In what context will you be teaching the denseness property to children?
3. Explain the concept of 1-1 correspondence.