Chapter 3 Whole Numbers

3.1 Numeration Systems

A common unit in third and fourth grade is the study of ancient number systems such as Egyptian or Mayan? As future elementary teachers you will study them because you may teach these systems to children. However, you will also study these systems to learn what is advantageous or not advantageous about them and how they relate to our system of numbers, the Hindu-Arabic System.

From Silver Burdett Ginn, Third Grade (New Jersey: Simon & Schuster, 1998) p. 41.

The Need for Numbers

Several years ago a two-part PBS series featured a primitive, nomadic tribe, which gathered sweet potatoes in jungles of Papua, New Guinea. Their numbering system consisted of **1, 2, and many**. One, two, and many were their only numbers. Why didn't they have more numbers? They had no need for other numbers. In their daily lives they could talk about one sweet potato, two sweet potatoes, or many sweet potatoes!

Numbers arose out of a need! The first person that balanced a checkbook probably realized that they had a need for negative numbers! Numbers were not just created whimsically.

The Roman Numeral System also has practical uses in everyday life. What are some examples of where Roman Numerals are used?

Advantages of Our Number System

Some advantages of our system that we can compare and contrast with other systems are:

1. Base 10 – Like the Egyptians
2. Place Value – Like the Babylonians
3. 10 digits – A big improvement over systems with only two or three symbols and systems with many symbols
4. Zero – Note that, like the tribe in Papua, New Guinea, children start with 1, not 0. Zero was also one of the last numbers developed by man. Zero is used as both a placeholder and a quantity (nothing) on which operations can be performed.

Why do we have base ten rather than base eight? In all likelihood, if aliens from outer space came to this planet and had 8 fingers, they would use base eight or sixteen, not ten!

Models of Our Number System

A. Unifix Cubes (a brand name also known as linker cubes, multi-links etc.)

This is a great model for young children especially if they are kept in strips of 10. Unifix Cubes may help children make sense of borrowing and carrying, also called trading (the politically incorrect term), in the standard addition and subtraction algorithms. Consider the process of carrying or trading in 27 + 38.

$$\begin{array}{r} {\scriptstyle 1} \\ 27 \\ \underline{+38} \\ 5 \end{array}$$

A child may represent 27 with 2 strips of ten cubes and 7 loose cubes and 38 with 3 strips of ten cubes and 8 loose cubes. Then 10 of the loose cubes may be stacked together to form another strip of ten. In other words, the child puts 10 ones together to make 1 ten.

Thus, the child has 3 + 2 + 1 tens (strips) and 5 ones (individual cubes). One advantage of unifix cubes over other manipulatives is that they allow children to physically stack 10 ones together to make 1 ten. Or, in the case of borrowing, children may break a ten into 10 ones. The experiences of “making” a ten and “breaking” a ten into ones are crucial in helping children come to see ten as both 10 individual units and as a unit itself (which is composed of 10 smaller units). Children need to be able to move flexibly back and forth between these two conceptions. Therefore, a strip of 10 Unifix Cubes is a good manipulative because it can be thought of as 10 cubes or as 1 strip of 10.

B. Base Ten or Dienes Blocks

Base Ten Blocks consist of the following:

1. A unit represents 1.
2. A long represents 10.
3. A flat represents 100.
4. A block represents 1,000.

From Silver Burdett Ginn, Third Grade (New Jersey: Simon & Schuster, 1998) p. 86.

Three considerations before using Base Ten Blocks are:

1. Foremost, the units are very small and young children may swallow them. Base Ten Blocks are also too small for young children to manipulate easily.
2. Next, consider the block, which is supposed to represent 1,000. What is a common misconception that children have about this manipulative? Often children conceptualize the block as representing 600 because it has 6 faces of 100 each. They cannot visualize the center flats. (The new plastic Base Ten Blocks that Velcro together may be helpful for this but few schools will have this newer, more expensive version of the manipulative.) In one classroom, a group of fourth graders was trying to convince a classmate that the block represented 1,000 but he could not see it and insisted the block was 600. More often than not the teacher will quickly go over the values of the manipulatives and then spend more time illustrating computation problems with them, not realizing that their may be a few children who are very confused with the illustration because they are thinking the blocks represent 600.
3. Base Ten Blocks are very common in elementary schools. However, the teacher will rarely have enough for the entire class to use. Thus, it is often the case that Base Ten Blocks are used by the teacher as a physical model, but they are not used as manipulatives. **A manipulative is something that a child manipulates.** However, Base Ten Blocks may be useful as a remediation tool with a smaller group of students.

Expanded Notation

This is a common activity that children will do in 4th grade on up. Why are children asked to do this? The purpose is that in a number like 576 we want children to know what the 5, the 7, and the 6 represent. However, children may just do the task of expanded notation, following the pattern, without developing a real understanding of place value.

$$576 = (5 \times 100) + (7 \times 10) + (6 \times 1)$$

A type of questioning that might help children construct viable understandings of place value is to ask, "How many tens are in 576?" Questions such as this focus more on the meaning of the digits than simply asking, "What digit is in the tens place?" Changing numbers back and forth between expanded notation and standard notation is a common activity in elementary textbooks.

Palindromes

A palindrome is a word or number that is the same frontward and backwards. What are some words that are palindromes? Bob, mom, dad, pop, level, racecar, radar, etc. Is "I" a palindrome? Some numbers that are palindromes are: 121, 22, and 48,284. Is "7" a palindrome?

Homework: Find all the palindromes from 1,000 – 2,000.

Consider the following assertion: every number can eventually be made into a palindrome by the process of reversing the digits, adding the original and new number, and repeating if necessary. For example, 23 is not a palindrome but if the digits are reversed and added, 23 + 32 = 55, which is a palindrome. Twenty-three took one time. Try 37. Reverse the digits and add, 37 + 73 = 110. The sum, 110, is not a palindrome but if the process is repeated, 110 + 011 = 121, which is a palindrome. So 37 takes two times.

Homework: Find numbers that will take 1, 2, 3 and more times. (Stick with double-digit numbers.)

Try 98.

The number 98 takes between 20 and 30 times. This is a great problem to give to 5th and 6th graders to practice addition. They can start with a calculator but the calculator soon is ineffective, as the numbers grow to be more than the number of digits possible on the calculator.

From Math Central, Third Grade (Boston: Houghton Mifflin, 1998) p. 51.

3.2 Addition and Subtraction

Addition and subtraction are defined in terms of set theory but is this how children think?

How might children in kindergarten or first grade who do not know the fact solve 3 + 4?

Children will initially count.

- They may count using Unifix cubes or counters.
- They may count on their fingers
- They may count in their heads.

In all these cases they are counting.

How will the same children solve 7-5? They will count!

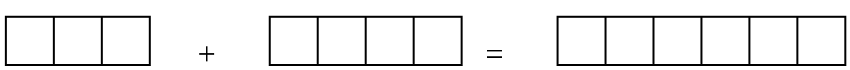
Since both addition and subtraction are counting activities for young children, it may **not** be necessary to separate problems into addition and subtraction as many elementary textbooks do.

Some textbooks attempt to classify addition into different categories such as adding measures and adding sets. However, young children will solve both types of problems by counting.

One Child's Perspective of Addition

Addition is frequently explained to children as the operation of combining or putting things together. One college student, in a mathematics methods course for elementary teachers, told this story of how she interpreted her teachers' instructions for addition. To solve 3 plus 4 she thought of a strip of 3 and a strip of 4. How do young children physically combine things? They use tape or paste. If you use paste you would overlap one of the squares of the 3 and the 4, which would give you 6. This is how she said she thought of addition. The class asked her why didn't she always get the wrong answer and know that something was not right. She said that she knew her answer was always 1 less than what the teacher wanted so she always added 1 to her answer.

This illustrates the importance of asking children to explain their thinking regardless of whether their answers are correct or incorrect. How would a teacher ever know that a child was thinking this way unless she asked the student how she solved the problems?



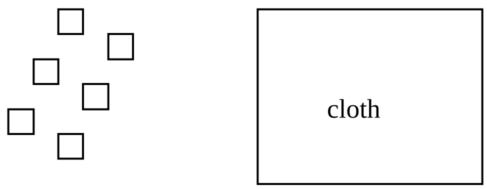
Counting Types

Research suggests that all children develop certain types of understanding as they develop mathematically. That is, they seem to pass through the same levels of development, although they do not all progress at the same rate. For counting, five different levels of development, or five different "counting types," have been identified.

They are the following: Perceptual, Motor, Verbal, Abstract, Part/Whole (Steffe, von Glasersfeld, Richards, & Cobb, 1983).

Perceptual Counters

Children at the perceptual counting stage need to see and touch objects in order to count them. They cannot count hidden objects. For example, they would be unable to solve a problem like that below where they are shown some squares, told that three more are hidden under the piece of cloth, and asked how many there are altogether.



Perceptual counters need to make collections of objects to count. They cannot count abstract objects. In other words, when perceptual counters use their fingers to count, their fingers are the objects they are counting. The number words they say refer to their fingers. They cannot use their fingers to keep track of the counting of abstract objects (e.g., imagined squares, days of the week).

Consider a word problem such as “Josh had 5 marbles. Joey gave him 4 more. How many does he have now?” If you put out 5 marbles and a pile of marbles, perceptual counters will take the whole pile and add it to the 5 rather than counting out 4 from the pile. While this may seem odd from an adult’s point of view, these children are simply expressing their concepts, that is, what they know and how they think. For perceptual counters, a number word such as “four” just refers to a bunch of objects. For them, the number 4 is not really there until they count 4 objects they can see.

Motor Counters

Like perceptual counters, motor counters have to “make” a number. A number is not real for them until they count it. Motor counters do not have to touch objects in order to count them and they can count hidden objects, but they must point with their fingers or use some type of sensory-motor action to count. Motor counters can solve the problem with visible and hidden squares described above, but they do not do it by saying, “six … seven, eight, nine,” that is, by counting on from 6. Instead, they must start counting at 1 in order to make the 6. Then they will count the three hidden squares by pointing at three imagined squares. If the above problem was changed so that children were told that there were some squares hidden under the cloth and that altogether there were 9 squares, motor counters would not be able to determine the number of hidden squares. This type of problem, called a “missing addend” problem, is conceptually more difficult because it requires the ability to count abstract objects as explained in the section on abstract counters below.

Verbal Counters

Verbal counters are very similar to motor counters in their developing abilities. The primary difference is that verbal counters do not need to make a motor action as they are counting. However, they still need to count from 1 and they cannot do missing addend problems. The most sophisticated problem they can do is one such as 9 – 4. They will put up 9 fingers, counting as they do so, and then take four away. In other words, children at this level can make collections and take away from them by counting. Then, in general, they will count to determine how many are left. Like perceptual and motor counters, verbal counters do not have an abstract concept of number. For them, a number is meaningful only when it refers to a specific collection of objects they have counted. They cannot just put the number 7 in their heads and think that it represents 7 objects of some kind. They need to think of actual objects and count "One, two, three, four, five, six, seven." This is what we mean when we say that they need to "make" a number.

Summary and Discussion of Pre-Abstract Counting Types—Perceptual, Motor, Verbal

Perceptual:	must actually see objects in order to count them	· cannot do missing addend problems · always begin counting at 1 · do not have an abstract concept of number, i.e., a number is not real until they make it by counting actual objects
Motor:	can count hidden objects; points at them with finger	
Verbal:	can count hidden objects; does not need to point	

It is important to realize that children at these levels are developing normally. In kindergarten, most all children are perceptual, motor, or verbal counters. About one-half of beginning first graders are at one of these levels and a few second graders may be also. Most students move beyond verbal counting during first grade. Instruction for children at these pre-abstract levels needs to include activities involving spatial number patterns (e.g., dot patterns, finger patterns, ten frames) and counting.

Finger Counting

A related issue that arises in the early grades is the use of finger counting. Should teachers discourage finger counting? For children at these pre-abstract levels, counting objects is the only way they have of solving problems involving numbers. If finger counting is prohibited and no other manipulatives are available, some students may use their toes or the numbers on a clock, but those who don't figure out such clever ways will be lost. Thus, it is preferable to keep finger counting out in the open. The teacher needs to see what methods children are using. Identifying a child's methods can help a teacher determine the level at which a child is operating. Knowing what level a child is at is important for teaching because it enables the teacher to select developmentally appropriate activities. As noted, banning finger counting and drill-and-practice on the basic facts will be detrimental to children at these pre-abstract counting levels. Instead, these children need activities that will permit counting while simultaneously challenging them to develop more sophisticated means of thinking about number.

Abstract Counters

The ability to solve missing addend problems (e.g., 9 visible squares, some hidden, 13 in all, how many are hidden?) is evidence that children have reached the abstract counting level. Typically, they solve such problems by counting on. For the above problem, they might count "nine … ten, eleven, twelve, thirteen," putting up one finger for each of 10, 11, 12, and 13. Abstract counters can also solve subtraction problems by counting backwards. For 14 – 5, they might count "fourteen … thirteen, twelve, eleven, ten, nine," again putting up one finger for each of 13, 12, 11, 10, and 9. There are two important differences between this type of counting and that evidenced by pre-abstract counters. First, abstract counters can put a number (at least a relatively small number) in their head and work with it meaningfully. They do not have to count it out starting at 1. Second, notice that both of the examples above involve a type of "double counting." For example, in the missing addend problem, the child says "ten, eleven, twelve, thirteen" while putting up 4 fingers and uses this to determine that the answer is 4. This indicates that the child is using her fingers as a record of her counting. She is not counting her fingers as objects, as a pre-abstract counter does, but instead is using her fingers to keep track of the counting of abstract objects.

Part/Whole Counters

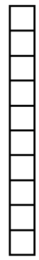
Part/Whole counters move one step beyond abstract counters in that they can solve subtraction problems in two different ways. Not only can they count backwards to solve subtraction problems, as abstract counters do, but they can also "close the gap." In other words, they can solve a problem such as 16 – 13 by counting "thirteen … fourteen, fifteen, sixteen," putting up one finger for each of 14, 15, and 16 to determine that the answer is 3. This capability indicates that part/whole counters see a relationship between addition and subtraction. It also means that they are beginning to form an understanding of part/whole relations. In order to be able to close the gap in the above problem, a child must view 13 as part of the whole collection of 16—the child then counts on to find the missing part.

Children at the abstract and part/whole counting levels need activities that will help them move beyond counting, develop thinking strategies (discussed following the next section), and begin to learn some of the "basic facts." Part/Whole Counters have developed the concept of ten - that is - they can think of 10 as a whole or chunk of 10 and they can think of 10 as parts, consisting of 10 units. Significantly they can go back and forth between the two conceptions.

Concept of Ten and Place Value

Children initially view 10 as ten separate units. A child with this conception of 10 can only solve a problem like 25 + 16 by counting on from 25 by ones. Next, children come to see 10 as either ten separate units or one single unit, but not both at the same time. In other words, they cannot coordinate these two different ways of thinking about 10.

Children at this level can count by tens, but they cannot count on by ten from a given number unless that number is 20, 30, 40, 50, etc. At the next level, children come to see 10 as a unit that is composed of ten smaller units. This is a part/whole conception of 10. They can think of numbers as made up of tens and ones. As a result, they can count on by 10 from a number such as 38. Children at this level can coordinate the counting of tens and ones and can switch flexibly between the two. We could illustrate their mental representation of 10 by the following figure.



10 as a unit composed of ten smaller units

Research indicates that in order to develop this part/whole conception of 10 children need experience building and "unbuilding" tens. This may be done initially by using Unifix cubes or Multilinks that allow children to put individual cubes together to form bars of 10 and to break bars of 10 into individual cubes (ones). Note that base 10 blocks and activities using money do not provide the opportunity for students to physically make and break apart tens. This is not to say that such materials are not useful, but for some children the experience of actually building "a ten" with the cubes may be especially significant.

Thinking Strategies and Learning "Basic Facts"

Once children reach the abstract counting level, they are ready for activities that will help them move beyond counting and develop thinking strategies. A thinking strategy involves using a known result to figure out an unknown result. For example, a child might say: I know that 5 + 5 = 10, so 5 + 6 = 11.

Children tend to learn "doubles" such as 2 + 2, 5 + 5, etc. first, so their initial use of thinking strategies tends to build off of these known results. Problems can be sequenced to promote the development of these strategies and teacher questioning may also encourage students to relate problems and thus develop thinking strategies. Note that when children use the strategies above they are no longer counting by ones to solve problems.

Developing thinking strategies helps children move beyond counting and facilitates the learning of the basic facts or number combinations because the use of thinking strategies involves constructing relationships among the basic facts. These relationships make the facts easier to learn because children see the facts as related to each other rather than as isolated bits of information. In addition, if a child does forget a fact, being able to apply thinking strategies will enable that child to figure it out instead of simply not knowing.

In sum, rather than moving directly from counting to memorization, teachers need to help children develop thinking strategies, because the use of thinking strategies

- helps children move beyond counting toward the learning of the basic facts
- helps children construct a network of relationships among the facts
- provides a basis for methods for adding and subtracting larger numbers
- helps children develop number sense, i.e., the ability to take numbers apart and put them back together in a different way
- promotes the belief that mathematics is supposed to make sense

Two Levels of Difficulty for Subtraction

A distinction is made for subtraction, not because of the way children solve the problems, but because one type of subtraction problem is conceptually more difficult for children. In the following two word problems, the operation and number sentence are the same: 17 – 9 = 8.

1. Mary has 17 marbles. She gave 9 to her brother Tom. How many marbles does Mary have now?

2. Mary has 17 marbles. Her brother Tom has 9 marbles. How many more marbles does Mary have than Tom?

What is happening in each problem? That is, what is the physical action suggested? In the first problem the action is to take something away and it is classified as a **take-away** problem. In the second problem, two quantities are being compared and it is classified as a **compare** problem.

Which problem is more difficult for children, take-away or compare?

If these problems were given to second graders they would have more trouble with the compare than the take-away. We can understand this phenomenon in terms of the counting types discussed in the previous section. Children at the Part/Whole level can solve the compare problem because they can conceptualize 9 marbles as part of the whole collection of 17 marbles and then count on to find the missing part. However, children who are not yet at this level have great difficulty making sense of the compare situation because they can only think in terms of adding to or taking away from a given amount. They cannot mentally partition that amount into subsets and then compare one of the subsets to the whole.

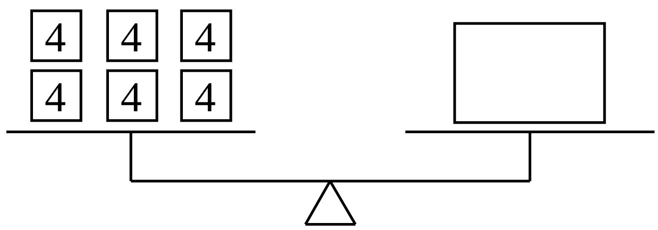
These two classifications for subtraction are made because they are different for children in terms of their difficulty! This is true even though children may still use counting to solve both types of problems.

3.3 Multiplication and Division

Multiplication

As children develop an understanding of place value and a part/whole conception of numbers (i.e., the ability to think of a given number as a unit which itself is made up of smaller units), they are forming a basis that will enable them to make sense of multiplication and division.

In elementary school, multiplication is typically presented as repeated addition. For example, on the balance problem below, children typically add 4 + 4 + 4 + 4 + 4 + 4 to get 24 and are told that a "shorthand" way to write this is 6 × 4 = 24.



From our adult point of view, we see this as a statement that 24 can be thought of as being made up of six units of 4. However, research indicates that just because children can perform repeated addition (and write multiplication number sentences to describe their repeated addition) does not mean they are yet capable of multiplicative thinking. Multiplicative thinking grows out of additive thinking, but is more complex

According to Clark and Kamii (1996), about 45% of second graders and 64% of third graders exhibit some multiplicative thinking. However, only 49% of fifth graders displayed what these researchers called "solid multiplicative thinking." This suggests that the ability to think multiplicatively develops slowly. Furthermore, just as asking pre-abstract counters to memorize basic addition facts may hinder their development and lead to nonsensical mathematical behavior, so too will asking additive thinkers to memorize the multiplication tables. Additive thinkers will interpret multiplicative situations additively because that is the only way they can interpret these situations. Thus the multiplication table will not make sense to them because they cannot use known facts to figure out ones they don't know, as they can for addition facts. For example, if an additive thinker is asked to use 4 × 4 = 16 to help figure out 5 × 4, she will probably reason that the answer is 17 because 5 is one more than 4.

Introducing Multiplication

Additive thinkers need to learn the number word sequences for counting in units other than 1. To this end, a common elementary-school activity is to have children count by twos, threes, fours, etc.

Of course, children also need an opportunity to relate multiplication ideas to "real-world" contexts. Simple word problems like the following can provide such an opportunity:

> Jennifer brought cupcakes to school. She had 6 cupcakes in each box. She had 5 boxes. How many cupcakes did she bring altogether?

Although, it is important initially for children to be able to "act out" problems of this type using objects of some kind to facilitate their thinking and counting, the teacher should encourage students to figure out the total "without counting by ones."

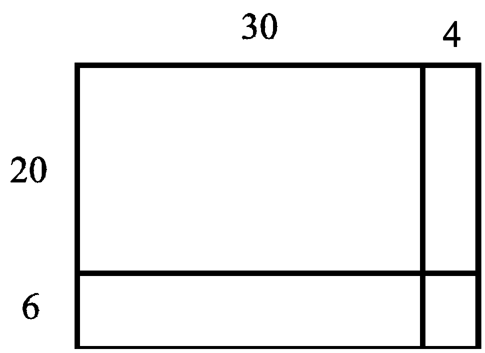
Cartesian products can be used to illustrate multiplication but they represent a more complex view of multiplication than the repeated addition view. An example of a Cartesian product problem is: If I have 3 pairs of pants and 4 shirts, how many different outfits can I make? Cartesian products are important later in probability and in the counting principle but this may not be a good way to introduce children to the concept of multiplication.

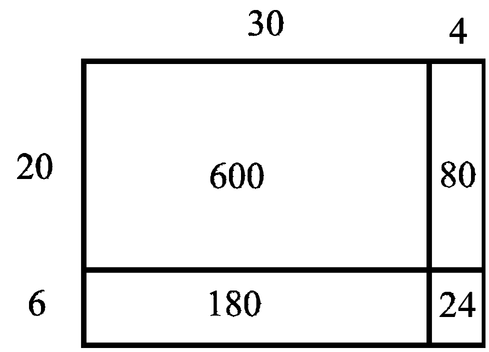
Multiplication Thinking Strategies and Basic Facts

As children begin to employ multiplicative thinking, it is important to encourage them to develop thinking strategies that will facilitate their learning of the multiplication basic facts ("multiplication tables") and enable them to construct a network of relationships among the facts. One example of a multiplication thinking strategy is "I know 5×5 is 25, so 6×5 must be 30 because it is just <u>one more 5</u> than 5×5." Another example is "I know 8×5 is 40, so 8×6 must be 48 because to get 8×6 you <u>add one to each of the eight fives</u> in 8×5 so 8×6 is 8 more than 8×5." As the examples illustrate, these thinking strategies involve multiplicative thinking, that is, the ability to simultaneously think about units of one and units of more than one. Consequently, helping students to develop thinking strategies solidifies their multiplicative thinking and provides a foundation for learning the basic facts. One way to promote use of thinking strategies is to sequence problems (e.g., 2×3, 4×3, 5×3, 5×6, 5×7) and encourage children to relate a given problem to a previous problem or problems ("Could one of the problems we've already done help you figure out this one? How could it help?")

Introducing Two-Digit Multiplication

For students who have constructed the concept of area, (fourth or fifth grade),a pictorial illustration of multiplication may be useful to teach two-digit multiplication. For children who do not have concept of area this may not be an appropriate model. For 26 x 34, draw a box, labeling it 34 across and 26 down. Divide the numbers into tens and ones as shown, find the area of each section, and add up all the areas.





26 x 34 = 600 + 180 + 80 + 24 = 884   This also illustrates the distributive property. 20 x 34 = (20 x 30) + (20 x 4) and 6 x 34 = (6 x 30) + (6 x 4)
20 x 34 = 600 + 80   6 x 34 = 180 + 24
which is: 26 x 34 = 600 + 180 + 80 + 24 = 884

<u>How Young Children Divide</u>

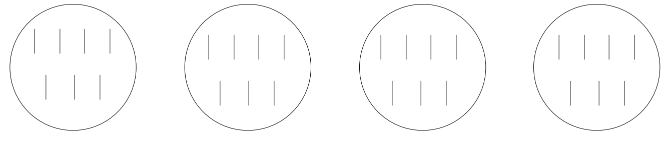
Your textbook may make several distinctions for division but this supplement only makes two because children who do not know their division facts and who do not yet know that multiplication is the opposite of division predominantly solve division problems in two distinct ways.

As early as second grade, children are capable of developing solution strategies for problems such as the following:

A. Mrs. Wright has 28 students in her class. If she divides them into 4 groups, how many students will be in each group?

B. Mrs. Davies has 30 students in her class. She wants to divide them into groups of 5. How many groups will she have?

Initially, students solve these problems using physical materials (e.g., cubes) or pictures to model the situation. For instance, for Problem A, students might draw 4 circles and sequentially allocate tally marks to the circles (i.e., one in this group, one in this group, one in this group, and so on) until they have made 28 tally marks. Then they can count the number of tally marks in each circle.



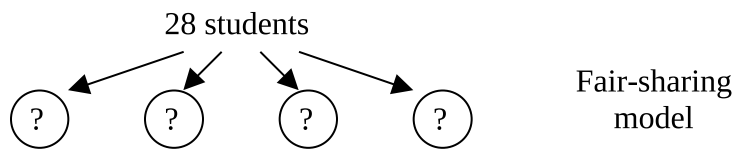
(As children's methods become more sophisticated they might allocate more than one at a time. For example, they might allocate 5 to each group on the first pass, then one to each group, then one more to each group.)

For Problem B, students might make 5 tally marks, circle them, make 5 more tally marks, circle them, and so on until they have made 30 tally marks. Then they can count the number of groups of 5 they have made. Alternately, students might add 5's until they get to 30 or subtract 5's until they get to zero, i.e.,

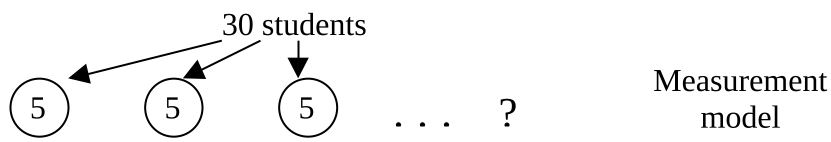
> 5 + 5 = 10, 10 + 5 = 15, 15 + 5 = 20, 20 + 5 = 25, 25 + 5 = 30 or
> 30 – 5 = 25, 25 – 5 = 20, 20 – 5 = 15, 15 – 5 = 10, 10 – 5 = 5, 5 – 5 = 0.

Then they can count the number of 5's they added or subtracted.

The two problems above illustrate two different interpretations of division. Note that Problem A involves forming a given number of groups. The problem then is to determine the size of each group. This is called the **fair-sharing** or **partitioning** model of division.



In contrast, Problem B involves forming groups of a given size. The problem then is to determine how many groups of that size can be formed. This is called the **measurement** or **repeated operation** model of division.



The reason this is called a repeated operation model is obvious from the above solution examples—students repeatedly add or subtract the size of each group. The reason it is sometimes called a measurement model is that both the approach of repeatedly adding or subtracting 5 and the approach of making 5 tally marks, circling them, making 5 more tally marks, circling them, etc. are analogous to successively laying 5-unit rulers end-to-end until the entire 30 units is "measured."

Young children may at first be more comfortable with problems that fit the fair-sharing model because they are likely to be familiar with the activity of sharing. However, it is important to ensure that they have many experiences with both types of division situations. The context of the problem plays a large part in determining the method of solution that children will employ.

With experience, children's solution strategies become more sophisticated. For the problem

Tonight 45 parents will be visiting our school. Six parents can sit at each table. How many tables do we need?

the variety of solution strategies typically produced by second and third graders might include the following:

1) using cubes or drawing a picture to model the situation
2) using repeated subtraction (i.e., 45 – 6 =39, 39 – 6 = 33, etc.) and keeping track of the number of 6's subtracted
3) using repeated addition to build up to 45
4) using a known multiplication fact as a shortcut and building up to 45 from there (e.g., 6 × 6 = 36, plus another 6 is 42   7 tables will seat 42 so altogether 8 tables are needed)

Of course, the above problem does not work out "evenly" and it is important for children to experience this because such problems are common in everyday situations and children need to think about how to deal with the "remainder." For example, in the above problem the remainder necessitates an additional table. However, if the problem involved sharing 45 marbles among 6 children, the 3 "leftover" marbles might be ignored. And for a problem of sharing 10 chocolate bars among 4 children, it might well be possible to split the 2 leftover chocolate bars among the 4 children so that each gets $2^{1}/_{2}$ bars. The point is that children should be encouraged to construct answers that make sense in the context of a problem. To say that $7^{1}/_{2}$ tables are needed for the parent meeting is nonsense and to say "7 remainder 3" does not fully address the problem.

Division By and With 0

Fifth graders may experience division by and with zero. Many times teachers just **tell** students that division by 0 is not possible without explaining why. Preservice teachers should be able to explain division by and with 0 to fifth graders.

7 ÷ 0 = ?   0 ÷ 7 = ?

Most fifth-grade students know that multiplication is the opposite of division and many of them use this knowledge to solve single-digit or two-digit division problems. For example, to find 56 ÷ 7 = ?, they often ask themselves 7 x ? = 56.

To explain division by and with 0 to children, write out a division number sentence and underneath it write out the corresponding multiplication sentence.

For example,

12 ÷ 3 = ?	7 ÷ 0 = ?	0 ÷7 = ?
3 x ? = 12	0 x ? = 7	7 x ? = 0

For 0 x ? = 7, there is nothing that you can multiply by 0 to get 7 so the problem is not possible or undefined. For 7 x ? = 0, one can replace the ? with 0 so this answer is 0. Not all fifth graders may understand this explanation but it is important that they see that there is a reason for the rules in mathematics and that it is not magic! Some children will understand the explanation.

<u>Order of Operations</u>

While your textbook covers order of operations it is not likely that you will be teaching all of the rules at once to children. You are more likely to be teaching the order of operations in regards to addition and subtraction for third grade, and the order of operations in regards to multiplication and addition/subtraction for fourth and fifth grade. In fifth grade you might also introduce grouping symbols.

Solve each of the following problems twice, using a different order of operations each time. Does the order of operations make a difference in the solution? In each problem, identify which of your orders corresponds to the rules for the order of operations. Can you explain to children why we need rules for the order of operations?

7 + 5 – 2	6 x 4 ÷ 2	6 x 3 + 2
9 – 4 + 3	18 ÷ 2 x 3	24 ÷ 4 – 2

From Everyday Mathematics, Grade 5 (Chicago: Everyday Learning, 1999) p. 226.

3.4 Properties of Whole-Number Operations

Even algebra students have a difficult time with tests requiring them to name the properties. This is puzzling considering that high school students intuitively know the properties when they involve numbers but not variables. They know intuitively that 3 + 4 = 4 + 3, but they have difficulty naming the property illustrated by the abstract statement $a + b = b + a$. Understanding how children come to know these properties can help explain this phenomenon.

Commutative Property

Addition

In the textbook the commutative property for addition and for multiplication is defined and explained. To illustrate how children think about the commutative property of addition, try the following activity, called "Target." This activity is done with elementary students for the simple purpose of practicing computation. In the activity of Target students are to give two numbers that give the target number—in this version they may only use one operation, addition. As a suggestion, pretend that you are first or second graders. Let's start with the target number "10." Give pairs of numbers whose sum is 10.

Did your class give both 7 + 3 and 3 + 7? Would first and second graders give the reversed pairs of numbers? Why?

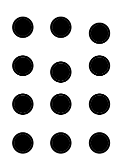
Research shows that addition is a counting activity for young children. So the problems 7 + 3 and 3 + 7 are extremely different for children if they are counting on. For 7 + 3, the child might say, "7 … 8, 9, 10," holding up a finger for each of 8, 9, and 10. For 3 + 7, the child might say, "3 … 4, 5, 6, 7, 8, 9, 10," holding up a finger for each of 4 through 10. For young children addition is not commutative. In this example, 7 + 3 and 3 + 7 are two different problems for children because they solve them differently. They do not see beforehand that the answers will be the same. The commutative property is a concept and children will not be able to make sense of it until they are ready.

However, as early as second grade, some children do figure out that 4 + 3 gives the same sum as 3 + 4, 5 + 8 gives the same sum as 8 + 5, etc. From an adult perspective, it is tempting to say that such children understand the commutative property of addition: For any two numbers a and b, $a + b = b + a$. We expect that they will be able to make sense of this formal statement of the property (and wonder why such a big deal is being made of it) when it is presented to them in an algebra class. However, closer examination of children's thinking about "commutativity" suggests that although they may view the equality of certain "turn-arounds" (e.g., 3 + 4, 4 + 3) as a fact, many are not certain that this would hold for any pair of numbers. For example, asked to find the sum 7 + 15, the same children who know that 3 + 4 = 4 + 3 will count on from 7 rather than reason that 7 + 15 = 15 + 7 and then count on from 15. Further, children's efforts to investigate the equality of turn-arounds for any pair of numbers reveal much about their developing understandings of addition. Some students check many pairs of numbers, often trying numbers larger than they usually work with. For these students, addition is a procedure applied to two numbers to get a result.

The important point here is that children do develop intuitive notions about commutativity, but they do not spontaneously construct the concept in full-blown richness of detail. They need a multitude of experiences with adding numbers before they intuit the commutative property of addition let alone name it. Their understanding can be furthered through activities that ask them to explore ideas related to commutativity and, in doing so, generate hypotheses, attempt generalizations, and construct justifications. It is this type of mathematical activity that will more fully prepare them for their later study of algebra.

Multiplication

In fourth grade most students will give the same solutions as adults would for the target addition activity and not give the reversed pairs. However, at this grade level, if the target operation were changed to multiplication many fourth graders would give reversed pairs of numbers. For example, for the target number 12 they would give 4 x 3 and 3 x 4. Through activities such as this children soon figure out that 4 × 3 gives the same result as 3 × 4. Of course, knowing that the result is the same is one thing; understanding why it is the same is quite another. The situation is complicated for multiplication by the fact that students are likely to think of 4 × 3 very differently than they think of 3 × 4. That is, they think of 4 × 3 as meaning 3 + 3 + 3 + 3 or four 3’s and they think of 3 × 4 as meaning 4 + 4 + 4 or three 4’sFor multiplication, arguing why four 3’s should be the same as three 4’s may prove more challenging. However, students who have related arrays to multiplication may reason that 4 × 3 gives the same result as 3 × 4 because the same array may be viewed as four rows of 3 dots or three columns of 4 dots.



Once some students figure out that multiplication is like addition in that 4 x 3 and 3 x 4 have the same solution, they often exert peer pressure on other students and argue against the reversed pairs. The teacher may see it their way as well. Consequently, many fourth-grade classes will not give reversed pairs for multiplication but this is not an indication that every student has figured out that multiplication is commutative. More importantly, it is not an indication that students understand why multiplication is commutative.

Associative Property

Students also intuit the associative property. Sometimes it is even discouraged by well-meaning teachers who insist that students must add from left to right and must multiply from left to right. The associative property can make problems easier and more meaningful for children. Give children problems that will encourage them to develop the associative property. For third or fourth graders, ask students to solve the following problem mentally.

57 + 88 + 12

Note it would be much more difficult to add 57 and 88 mentally but 88 + 12 is 100 and 100 + 57 is 157.

For the associative property of multiplication give a problem such as the following:

17 x 25 x 4

Again note that it would be difficult to solve 17 x 25 mentally but it is much easier to multiply 4 x 25, which is 100, and then 100 x 17 is 1700!

As a teacher you can give students several problems like these to encourage them to construct the associative property for addition and multiplication.

Distributive Property

The distributive property is very important in algebra.   Unfortunately it is typically not presented in a meaningful way to elementary students. A typical problem involves having children apply the distributive property to compute a product in two ways.

4(17 + 3) = (4 x 17) + (4 x 3)
4 x 20 = 68 + 12
80 = 80

Students do not see the need to solve the problems on the right when it is easier to solve the problem without distributing the numbers. Children can develop intuitive notions about the distributive property if they are presented with the appropriate multiplication activities. Some of these are illustrated and discussed in the next section.

Identity Properties

Children think it is silly when you tell them these properties. They know that any number plus 0 is the number and any number times 1 is the number. From a child's perspective, why is something so simple given a fancy name?

Closure Property

As future teachers you will probably never teach this property to children but it is a property that drives our creation of numbers. That is, we create new numbers so that we have closure. For example, addition of whole numbers is closed because the sum of any two whole numbers is a whole number but subtraction of whole numbers is not closed (e.g., 2 – 5 = −3). What numbers are subtraction and the closure property going to cause us to create? Negative numbers.

From Math Advantage, Grade 5 (Orlando: Harcourt Brace, 1998) p70.

Algorithms

Most elementary textbooks teach "standard algorithms" in great depth. An algorithm is a series of steps or procedures that are repeated. The standard algorithms for addition and subtraction involve putting the addends in columns, with each column corresponding to the appropriate place value. The addition algorithm involves "carrying" a number to the top of the next column when the sum is more than 10 and the subtraction algorithm involves "borrowing" when the bottom number is larger than the top number in a column. Multiplication also involves putting the numbers in columns and multiplying by each digit and adding the products. The standard algorithm for division is called "long division." There has been a debate about whether we should spend a lot of time on long division when in real life most people will use a calculator to solve division problems with larger numbers.

Addition and Subtraction Algorithms

Overview

It took mankind thousand of years to develop these standard algorithms and even though they are efficient they are not the only way to compute. As an illustration, consider the following subtraction algorithm that goes from left to right. Your grandparents or parents may have learned it.

3,458
-1,769
~~2~~,~~79~~9
1,689

This algorithm starts on the left. First, subtract 3 − 1 (Subtract one thousand from 3 thousand) and write down the 2 (thousand). Go to the next column where you cannot subtract 7 (hundred) from 4 (hundred) so borrow one (thousand) from the 2 (thousand), crossing it out and making it a 1 (thousand). Now you can subtract 7 (hundred) from 14 (hundred), which is 7 (hundred), so write down the 7 (hundred) and go to the next column. Here you cannot subtract 6 (tens) from 5 (tens) so borrow from the 7 (hundred), making it a 6 (hundred), and then subtract 6 (tens) from 15 (tens), which is 9 (tens). In the last column you cannot subtract 9 (ones) from 8 (ones) so borrow one (ten) and cross out the 9 (tens), making it 8 (tens). Then 18 (ones)– 9 (ones) is 9 (ones).

This is an efficient algorithm, which will always work; it is just not the "standard algorithm."

Even though children practice the standard algorithms they often do not understand what they are doing. Mathematics is not meaningful.

```
 1
 27
+28
 55
```

Ask children what the *1* means when they carry. Often they say that this is how you do it or that is the way that they have been taught without indicating that the *1* really represents 10.

The standard algorithms are one way to solve problems but they are not the only way. Adults tend to think of standard algorithms as the only way to compute because this is how they were taught. However, children will invent their own ways of doing computation when not directed to follow a particular algorithm. These children typically possess a greater understanding of mathematics and are efficient in their computation (Madell, 1985). On the other hand, children taught the standard algorithms for addition and subtraction before they have fully developed the concept of 10 typically experience severe difficulties in understanding mathematics. The following discussion of children's own strategies illustrates why this is so.

Children's Self-Generated Algorithms

Developing a part/whole conception of numbers and an understanding of place value facilitates children's development of methods for adding and subtracting two- and three-digit numbers. Conversely, presenting students with two- and three-digit addition problems in the appropriate contexts (e.g., with manipulatives) can help them begin to think of numbers as made up of tens and ones (or hundreds, tens, and ones). In other words, children's concept of place value and their procedures for computing with multi-digit numbers develop together.

Elementary school children invent a variety of sophisticated algorithms for addition and subtraction. For example, for the problem 37 + 24, a child might reason "37 + 10 = 47, 47 + 10 = 57" (or "37 + 20 = 57"), then "57 + 4 = 61

After children have developed their own algorithms, the standard algorithms for addition and subtraction will be much more meaningful. These are shown below as they are typically written in column format.

```
   1            6 1
   37           X2
+  24        -  24
   61           48
```

When children have developed a part/whole conception of ten and the ability to think of numbers as collections of tens and ones, they may be able to make sense of these standard algorithms, which rely on the notions of combining ones to make a ten and breaking a ten into ones. Still, learning these standard algorithms presents significant challenges for children. For one, the standard algorithms work from right to left, whereas children's natural tendency is to work from left to right (see examples of children's algorithms above). And if an attempt is made to teach children these algorithms before they have constructed a part/whole conception of ten and their own strategies, they are reduced to performing symbol manipulations that they don't understand. It is startling how many high school students cannot explain the meaning of the 1 above the 3 in 37 in the standard addition algorithm example above. This lack of understanding leads students to develop what are sometimes called "buggy," or error-prone, algorithms. A common error for the above subtraction problem would be to subtract the 2 from the 4 in the ones column and come up with an answer of 52. Research indicates that children who understand the algorithms they are using do not make these kinds of errors.

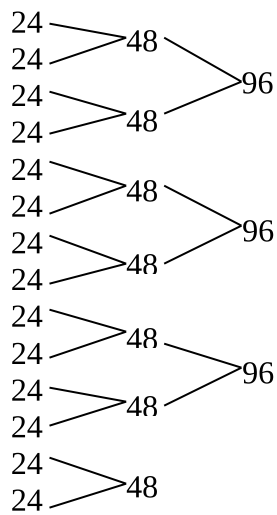
Textbooks sometimes try to explain the standard algorithms with pictures showing individual ones, sticks representing ten, and bundles of ten sticks representing one hundred. These pictures may help students who have already constructed an understanding of place value figure out what is going on in the standard algorithms, but they will be of little benefit to those students who have yet to construct such an understanding. The concept of place value cannot be apprehended from pictures. It develops from children's experiences with physical materials, from their own drawings representing numbers, and from their mental activity of combining smaller units to make a larger unit and breaking a larger unit into smaller units.

While a teacher may feel it necessary to guide students to use of the standard addition and subtraction algorithms, it is important to note that the role of paper-and-pencil calculation (for which these algorithms were created and are most helpful) is being reduced by the use of calculators. Furthermore, for the purposes of mental computation, the standard algorithms are rather clumsy and difficult to keep track of. Children's self-constructed algorithms tend to be much better suited for performing mental calculations efficiently and reliably.

Multiplication Algorithms

The standard algorithm for multi-digit multiplication that is typically taught in elementary school is a complex procedure that alternates steps of multiplying and adding and relies on the proper alignment of place values in order to produce the correct result. Although very efficient, the compactness of this algorithm hinders efforts to give meaning to what is going on in the various steps. Students may execute the steps without understanding how the ones, tens, hundreds, etc. fit into the manipulations they are performing (National Research Council, 2001). The difficulties stemming from this are the same as those that arise when addition and subtraction algorithms are learned without understanding: students develop "buggy" algorithms, they are lost if they forget a step because they have no other way to figure out the answer, and they cannot (or at least do not think to) assess the sensibleness of their answers.

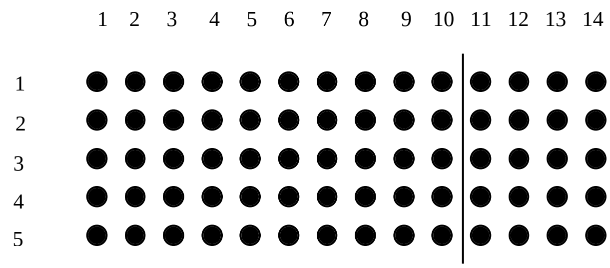
Research indicates that children are capable of inventing their own meaningful algorithms for multi-digit multiplication and these algorithms become increasingly sophisticated and efficient over time. When faced with the problem of determining the number of cans of pop in 14 cartons of pop with 24 cans in each carton, children may initially use repeated addition: 24 + 24 + 24 + …. However, this strategy is soon replaced by a more efficient "doubling" strategy that makes use of adding pairs of numbers, then pairs of pairs, as illustrated below (Baek, 1998; Caliandro, 2000).



$$\begin{array}{r} 96 \\ 96 \\ 96 \\ +\ \ 48 \\ \hline 336 \end{array}$$

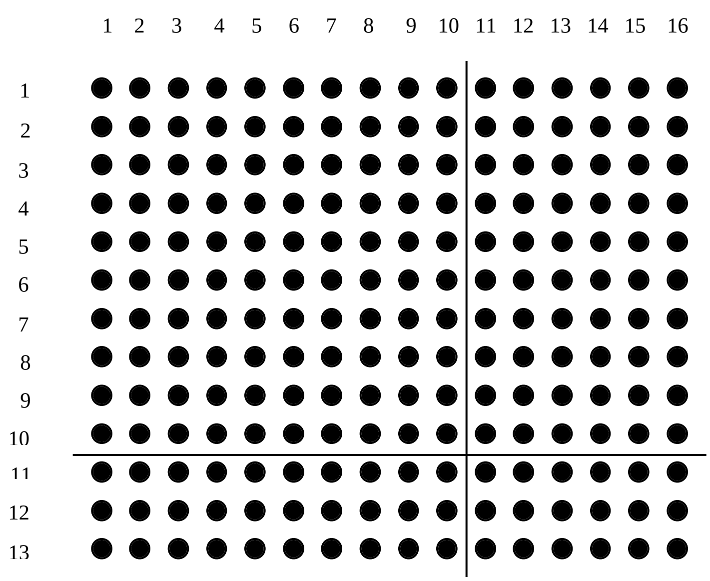
Next, students develop the strategy of partitioning one or both of the numbers in the problem, essentially breaking the problem into subproblems that are easier and/or allow them to apply multiplication facts they already know (Baek, 1998). For example, a student comfortable with multiplying two-digit numbers by one-digit numbers might compute 24 × 7 and then double that result (Caliandro, 2000). We could represent this solution by writing 24 × 14 = 24 × (7 × 2) = (24 × 7) × 2. Another example of a partitioning strategy would be the student who reasons that ten 24's would be 240 (because 24 × 10 = 240) and four more 24's would be 96 (because four 25's is 100) and then adds 240 and 96 to determine the total. Note that this last strategy involves splitting one of the numbers into tens and ones. Thus, it illustrates the important role that an understanding of place value and the ability to view numbers as composed of so many hundreds, tens, and ones plays in supporting children's construction of multiplication algorithms.

In addition to challenging students with word problems containing larger numbers, another type of activity that may help students develop their own meaningful multiplication algorithms and/or be able to make sense out of the standard algorithm involves using the overhead to show arrays of dots like that on the next page and asking students to determine the total number of dots.



Because the array is shown on the overhead screen, it is difficult for children to count the dots by ones so they are challenged to find more efficient ways to count them all. The vertical bar provides the opportunity for children to partition the dots in a way that takes advantage of place value. Some students may see the 5 by 14 array as consisting of 5 tens and 5 fours. In other words, the bar helps students partition the two-digit number into tens and ones in order to make counting all the dots much easier. This solution also makes implicit use of the distributive property of multiplication: $5 \times (10 + 4) = (5 \times 10) + (5 \times 4)$.

An array corresponding to the multiplication of two two-digit numbers is shown below.



Again, the bars provide the opportunity for partitioning the numbers into tens and ones and determining the "partial products" that correspond to this partition. By using their knowledge of place value, children can break the problem down into manageable parts. The algorithm below takes advantage of the partition suggested by the bars in the array and has been presented in some textbooks as a more accessible alternative to the standard algorithm (National Research Council, 2001).

$$\begin{array}{rl} 16 & \\ \times\ 13 & \\ \hline 100 & = 10 \times 10 \\ 60 & = 10 \times\ \ 6 \\ 30 & = \ \ 3 \times 10 \\ 18 & = \ \ 3 \times\ \ 6 \\ \hline 208 & \end{array}$$

Division Algorithms

As children develop a part/whole conception of numbers and the ability to think multiplicatively, presenting division situations involving larger numbers will help them begin to develop efficient algorithms for division.

Students may find the standard division algorithm taught in schools difficult for several reasons. First, the language we typically use with this algorithm suggests the measurement model of division, but not the fair-sharing model. That is, for problems such as $38\overline{)1296}$ or $5\overline{)231}$, we begin by asking "How many times does 38 go into 129?" or "How many times does 5 go into 23?" Second, in order for the algorithm to work, it is necessary to find the largest multiple of the divisor that is less than or equal the dividend. Third, these questions make it easy to lose a sense of the place value meanings of the numbers involved. For example, the 129 represents 1290 and the 3 we write above 129 really means 30 (National Research Council, 2001). Further, the question is not "How many times does 38 go into 1290?" but "How many tens times 38 go into 1290

From Silver Burdett Ginn, Grade 5 (New Jersey: Simon & Schuster, 1998) p. 217.

3.6 Mental Math & Estimation

Mental math and estimation are very useful in real-life applications. These skills also help children develop greater number sense. However, it is important to do mental math and estimation in ways that are relevant to children. What do you think many children do with the instructions on a sheet of computation problems that read, "**Estimate your answers first and then solve the problems**"? Many will simply solve the problems and then write an estimate based on their answer. From some children's perspective, it seems silly to estimate something when you are going to get an exact answer for it anyway!

One way to encourage number sense is by starting with simpler problems and moving to ones where children can use a thinking strategy to help them. Recall that a thinking strategy involves using a known result to figure out an unknown one. You can encourage the use of thinking strategies by sequencing problems as illustrated below. This will also help children develop number sense.

Third Grade	Fourth Grade
100 + 100 = ____	20 x 25 = ___
99 + 99 = ____	19 x 25 = ___
98 + 97 = ____	18 x 25 = ___

For estimation the most common school activity is rounding but in real life people do not always round according to the rules the textbooks specify. When people are grocery shopping and have a fixed amount of money, they tend to round up (e.g., 3.19 rounds to 4.00) to make sure that they have enough money. Similarly, when people are packing food for a hiking trip, they would often rather overestimate than underestimate.

A real-life application involving mental math and estimation is figuring the tip at a restaurant. How do you figure the tip at a restaurant?

In studies of how ordinary people use mathematics in their daily lives the context of the problem plays a large part in determining the solution (Lave,). When grocery shopping, what other factors besides price determine what and how much you buy?

Homework:

Don't use <u>paper</u> and <u>pencil</u> or a <u>calculator</u> to solve any of the problems. Estimate each problem and **describe how you did it**.

1. You are driving to Dallas and you have to take your 4-year-old niece and 7-year-old nephew with you. How long will the total trip take? Will you stop? How much will gas cost? What will you do to entertain your favorite relatives during the trip?

2. You are with a date that you want to impress. The bill is $37.45. How much of a tip will you leave? How did you figure it? Do you ask your date to pay for half?

3. You are going grocery shopping and you have a crisp twenty-dollar bill to spend. You need to buy milk, bread, toothpaste, cookies, Frosted Flakes, a dozen eggs, and a frozen pizza. Estimate your total. Do you have enough for a package of poki-mon cards?

4. How far do you live from the post office?

5. How often do you look at a clock in one day?

6. How far is it across a body of water that you are familiar with? How did you estimate this distance?

7. How long would it take to teach a fourth-grade class about Egyptian Math?

Chapter 3

Questions for Discussion and Review

3.1

1. Why did you and why do children study ancient number systems such as the Egyptian number system?
2. What are 4 advantages of our Hindu-Arabic number system?
3. Why do you think the tribe described in Papua, New Guinea, only had numbers for 1, 2, and many? What implications does this have for number systems we use?
4. Why are Unifix Cubes a good model of our number system?
5. Describe how you would use Base Ten Blocks to illustrate how to add 268 + 843?
6. What are 3 reasons why you might <u>not</u> use Base Ten Blocks?
7. How many tens are in 5,437? How many hundreds are in 5,437?

3.2

1. What process do children use when they first do addition? Subtraction?
2. What are two types of subtraction problems and which one is more difficult for children? Give an example of each.
3. What are the five counting types?
4. Is it okay for children to count on their fingers?
5. Why would you not give timed tests to pre-abstract counters?
6. What is the **concept of 10**?

3.3

1. What is multiplication for children first learning it?
2. Show and explain how you might use a pictorial representation to demonstrate two-digit multiplication.
3. What are the two ways that young children divide (assuming they do not yet know their multiplication or division facts)?
4. Explain: $5 \div 0 = ?$ $0 \div 5 = ?$
5. What are the order of operations and how can you justify the need for them to children?

3.4

1. Why isn't addition commutative for young children?
2. Give an example of how the associative property can make an addition problem easier.
3. Give an example of how the associative property can make a multiplication problem easier.
4. Why is the closure property important mathematically?
5. What do children think of the identity properties?

3.5

1. What are “standard algorithms”?
2. What does it mean to “carry a 1” when you are adding?
3. Should elementary school students spend a lot of time learning long division?
4. If children are not taught the standard algorithm to add and subtract, what will they do?
5. Do you think children are taught standard algorithms before they are ready to understand them?

3.6

1. What is a thinking strategy?
2. How could you use 100 + 100 = 200 to help you solve 98 + 98 = __.
3. How do you figure the tip at a restaurant?
4. When, in real life, would you round a number up even though the next digit is less than 5?
5. When you are grocery shopping, what other factors influence what you will buy besides estimating the price?
6. In elementary school, when you saw the instructions, “**Estimate your answers first and then solve the problems**,” what did you do? What do you think many children will do?

Chapter 4   Number Theory

Number theory is not explicitly taught in the elementary schools. However, many of the fundamental ideas of number theory are directly applicable to elementary school mathematics. Number theory is used to find the least common denominator when adding or subtracting fractions and to find the greatest common factor in reducing fractions. Number theory is often a graduate level course; however, the average person can frequently understand the mathematical ideas.

4.1 Factors and Multiples

The notation may be new and sometimes is confusing. Does 4½0 or does 0½4?

How will children find the factors of a number? How would they find the factors of 24? Typically, they will make a list of all the numbers that multiply together to make 24.

Likewise, to find multiples of numbers, children will make lists.

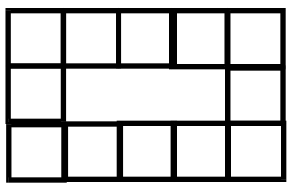
From Silver Burdett Ginn, Grade 2 (New Jersey: Simon & Schuster, 1998) p. 87.

A mathematical point that may be confusing is that 0 is the first multiple of every number. This may not be relevant when teaching children about multiples.

The following problem is designed to address the concept of factors. It is a good problem for elementary students, especially if they have square manipulatives.

What are all the ways that you can arrange 12 squares into a rectangular shape?

However, what if one student presents the following solution?



Would you count this solution? Does it meet the requirements of the problem? Does it alter the intended purpose of the problem, which was to teach children about factors?

This activity can also be used to illustrate the commutative property of multiplication (e.g., 3 x 4 = 4 x 3).

4.2 Divisibility Tests

Divisibility tests are taught in the upper elementary grades, especially the tests for 2, 5, 10, and often 3. The tests for 4 and 9 are usually not taught in elementary schools.

Solve the problem 6½23, 45_ . Give all the solutions.

How can you use the divisibility tests to solve this problem?

From Everyday Mathematics, Grade 5 (Chicago: Everyday Learning, 1999) p. 13.

4.3 Prime and Composite Numbers

Elementary students do touch on prime and composite numbers once they **understand multiplication and division**.

Homework: Make a Sieve of Eratosthenes from 1 – 100.

Elementary students may never use the prime number test but it is based on some significant mathematical ideas.

Is 701 prime or composite? To apply the prime number test, take the square root of 701 (26.48) and test all the primes up to 23.

Why are only prime numbers tested? Why test only the primes up to the square root of the number?

To answer the first question, if 6 were to go into a number, the numbers 2 and 3 must also go into the number. In other words, since every composite number can be expressed as a product of prime numbers, we only need to test prime numbers. For the second question, primes larger than 23 do not need to be tested because if a larger prime went into 701 evenly, for example 29, then 29 x ?? = 701 but the other factor, "??", would have to be less than $\sqrt{701}$ and all primes less than $\sqrt{701}$ have already been tested.

Number theory lends itself to some interesting true/false questions.

True or False: All prime numbers are odd.

An example of the accessibility of the mathematical ideas of number theory is in the movie "The Mirror Has Two Faces." The mathematician in the movie is attempting to prove that the number of twin primes [(3,5); (5,7); (11,13); (17,19); …] is infinite. Who plays the mathematician in this movie? One of the beauties of number theory is that the average person can understand the question and even though we cannot answer the question, neither can the mathematicians!

From Scott Foresman – Addison Wesley Math, Grade 4 (Menlo Park, CA, 1998) p. 185.

4.4 Greatest Common Factor & Least Common Multiple

Some elementary students may learn about prime factorization in fifth grade but it is a common middle-school activity.

Typically, children will find the Greatest Common Factor (GCF) of two numbers by making lists of each number's factors.

A common activity in elementary school is to make lists of the multiples of each number to find the Least Common Multiple (LCM) or the Least Common Denominator (LCD). For 7 and 8:

7,14, 21, 35,42, 49, 56, 63, 70, …
8, 16, 24, 32, 40, 48, 56, 64, 72, 80

While this method works well for smaller numbers it becomes very cumbersome for pairs of larger numbers like 210 and 144. Here the prime factorization method is more efficient.

The prime factorization methods for GCF and LCM could also be related to intersection and union in set theory. The GCF of two numbers is the intersection of the prime factors of the numbers and the LCM is the union of the prime factors of the numbers.

The following word problem addresses common multiples.

> **Two trains both begin their runs at 6:00 AM from the same station. Train A takes 60 minutes to complete its loop and Train B takes 72 minutes to complete its loop. When will both trains arrive simultaneously at the station provided that the trains are on time?**

This is a problem that could also be given to elementary school children. How would you solve this problem?

Another good problem is the Locker Problem.

> **There are 1,000 lockers in a school numbered 1 – 1,000 and 1,000 students. The first student goes through and opens every locker. The second student goes through and shuts every other locker (i.e., the lockers numbered 2,4,6, …). The third student goes through the school and changes the state of every third locker (that is, if the locker is open she shuts it and if the locker is shut she opens it), the fourth student either opens or shuts every fourth locker, and so on. Which lockers will be open when all the students are finished?**

How does this problem relate to number theory?

From Silver Burdett Ginn, Grade 5 (New Jersey: Simon & Schuster, 1998) pp. 336,338, 371.

Chapter 4
Questions for Discussion and Review

4.1

1. How will children find the factors of 40?
2. Would you teach children that 0 is the first multiple of every number?

4.2

1. Which divisibility tests might you teach in fifth grade?
2. Why does the divisibility test for 2 work?

4.3

1. What mathematical concepts should children have a good grasp of before they are taught about prime and composite numbers?
2. How many twin primes are there?

4.4

1. How will children find the LCM of 9 and 12?
2. How are prime numbers related to the GCF and LCM of numbers?
3. Will you be teaching prime factorization to children?