CMET
Connecting Mathematics for Elementary Teachers
Student Supplement

Chapter 1 Problem Solving

1.1 An Introduction to Problem Solving

<u>What is Problem Solving?</u>

What do we mean by "problem solving"? In this supplement, the term has two primary uses. On the one hand, it refers to what might be called "general problem solving"—the practice of engaging students in working on challenging, nonroutine problems (not necessarily word or story problems). The intent of such general problem solving sessions is to help students develop a repertoire of problem-solving strategies, which are sometimes called heuristics. On the other hand, the term "problem solving" may be used as a label for certain teaching practices encouraged by current reform efforts in mathematics education. In this case, we speak of "a problem-solving approach" to teaching or Problem-Centered Learning (PCL). PCL is characterized by a significant amount of small-group work and by whole-class discussions in which children explain their thinking, justify their solutions, and question each other. The teacher orchestrates the discussions, asking questions, challenging students' ideas, and offering guidance, but largely refrains from the traditional practice of showing students a single procedure for solving a certain type of problem and then having students spend a great deal of time practicing that procedure. Many topics, including general problem solving, multi-digit addition and subtraction, multiplication and division, concepts of fractions, spatial reasoning, and the notion of area, may be taught with such a problem-solving approach. In addition to its dual meanings in this supplement, be aware that problem solving can also have different meanings in the textbooks you will use. When encountering the term, it is a good practice to ask yourself, "What does 'problem solving' mean to these authors?"

Problem solving plays an important role in every elementary mathematics textbook from kindergarten to eighth grade. Some textbooks use "general problem solving" as a supplement or include several sections on problem solving, but others, which are often referred to as reform-oriented curriculums (e.g., Everyday Mathematics, Math Trailblazers, Investigations in Number, Data, and Space), integrate a "problem-solving approach" throughout the text and attempt to teach all topics through such an approach. More information on these reform textbooks is available at the Alternatives for Rebuilding Curricula (ARC) Center website: http://www.comap.com/elementary/project/arc/curricul.html. Whichever elementary mathematics textbook you use once you become a teacher, you will be teaching problem solving to your students! In addition, your future students will need to be good problem solvers in order to be successful in high school, college, and life applications.

Often students and teachers view mathematics as a collection of facts and rules and not as a sense-making activity. How are the preservice teacher's views given below similar or different from your own views of mathematics?

> Each following year, another layer of rules was added to the preceding year. This layer effect continued until one graduated from high school. Mathematics became a stumbling stone by the time I reached ninth grade algebra. In that freshmen algebra class I became convinced that I was not a "math student."

Teaching problem solving and employing a problem-solving approach to teaching can affect commonly held beliefs such as those described by the preservice teacher. In an appropriate classroom environment, problem solving may help students improve their self-image in relation to mathematics. Further, a problem-solving approach has been shown to help students develop a better and richer understanding of mathematics (Cobb, Wood, & Yackel 1991; Carpenter, Fenema, & Franke, 1996; Kamii & Housman, 1999).

Learning to Teach Problem Solving

What is the best way to learn how to teach problem solving? The best way is to experience problem solving. Therefore, as you solve the problems in your textbook and in this supplement, try to think about how this might relate to the problems that you will be giving children someday. Three questions that you might ask yourself, not just when doing problem solving, but throughout this course are:

- How might children solve this type of problem?
- What might children learn from working on this problem?
- Do I understand the mathematics enough to teach it to children?

Another preservice teacher said:

> I was taught to memorize rules and was tested with drills. It is common to teach in the same manner you were taught... I was ... very rule oriented in my thinking about math. It is a hard habit to break. I can remember thinking I just want to know the rule when learning math.

An essential question is how to break this recurring cycle of teachers teaching the way they were taught and emphasizing the memorization of rules. Memorizing rules without understanding does not serve students well when they need to apply mathematics to solve problems outside of school or when they need to apply their knowledge to learn more advanced mathematics. The NCTM Standards (1989, 1991, 2000) calls for:

> The creation of a curriculum and an environment, in which teaching and learning are to occur, that are very different from much of the current practice ... The kind of teaching envisioned in these standards is significantly different from what teachers themselves have experienced in mathematics classes. (pp 1-2, NCTM, 1991)

The intent of the activities in this supplement is to allow you to experience problem solving and enable you to teach problem solving to children. This will help you become a better teacher of problem solving.

What is a Problem?

There are many definitions but a more important question is, "What is a problem for children?" Two second-grade classrooms, one rural and one urban, were asked, "What is a problem?" For the children in the rural setting, an example of a problem was "if you miss the bus." The students were asked how they would solve this problem and they responded that you have to call your mom to take you to school. In the urban school, problems given by the children were "if you do not have enough to eat" or "there are gunshots at night." These are real examples from second graders. In this course you will be solving problems like that of the rural second graders—problems that you can solve. In real life not all problems can be solved.

In mathematics, what do you think a problem is?

A Mathematical Problem for Children

One of the major goals of this supplement is to help you learn to think like children do. So now let us examine an example of a mathematical problem for children.

An effective educational practice is to have children work together, especially when they are problem solving. Suppose that a teacher has decided to have children work in pairs. The teacher has the class count off beginning at 1. She tells them that 1 will pair with 2, 3 with 4, 5 with 6, and so on. (As you get to know your students better, it works well to pair students of like ability.) What if student number 17 raises his hand and asks, "Who is my partner?" If we are going to teach using a problem-solving approach and if we want our students to really do problem solving, then we cannot tell the students how to solve this problem. We must try to find a way to help <u>them</u> solve the problem. Also, if number 17 does not know who his partner is, it is likely that other students do not know who their partner is either.

Students in other mathematical content classes have suggested telling the children the person on their right is their partner—but some partners are sitting front-to-back, and others are on their left. Another suggestion is to tell them that if they are an odd number they should go to the next higher number. This will work, but in a first- or second-grade classroom student number 17 might respond, "I don't know if I am even or odd." How might a child determine if a number is even or odd? A common response is that they could divide their number by 2, but most first and second graders don't know how to divide by 2. Another suggestion is to redo the counting and count 1,2; 1,2; etc. but the teacher may not want to redo the process, as the teacher's goal may be to solve the problem without starting over.

How can you help number 17 and the other children in the class figure out who their partner is without telling them**? What makes this a problem for children? Is it the same or different from textbook problems given to children?**

<u>How Children Solve Problems</u>

Children are much more likely to act out or model problems than are adults. Adults tend to try to solve problems in their heads first through abstract thinking. Sometimes children's methods are more productive than adults' methods. Children are also more likely to use "trial and error" than are adults. Studies of expert and novice (which would include most children) problem solvers indicate that novice problem solvers tend to focus on the superficial information of the problem (National Research Council, 1985).

The following example occurred in a fourth-grade urban classroom with two girls who were working together on word problems. One of the problems asked:

How many pieces of candy could one buy for 72 cents if each piece costs 6 cents?

Initially, the two girls were stumped. However, with a great deal of thought and discussion, they made 72 (7 strips of 10 and 2 ones) with their unifix cubes and then divided the 72 into groups of six. They then counted the strips of 6 to arrive at their answer. Note how the girls had to physically model the problem with manipulatives in order to solve it. This is how children often solve problems.

The example on the next page from an elementary school mathematics textbook illustrates fourth-grade problem solving activity.

From Everyday Mathematics, Grade 4 (Chicago: Everyday Learning, 1999) p. 91

Problem Solving Is Not Just Word Problems

Often when students and teachers are asked to think of problem solving they think of word or story problems. Problem solving is not just solving word problems as the problem involving finding one's partner illustrates.

The next textbook example illustrates the type of problem solving in kindergarten.

From Silver Burdett Ginn, Kindergarten (New Jersey: Simon & Schuster, 1998) p. A13

Problems - Focusing on the Process

Try to solve the following problems. Focus on the processes you use. Also, consider how children might solve similar problems. One of the key suggestions for teaching problem solving is to focus on the process, not the product! In these problems the processes you use are what are important. If children understand the processes then they will be able to apply or transfer what they have learned to other problems.

1. Find the sum: 1 + 2 + 3 + ... + 998 + 999 + 1,000 = ?

   What patterns did you find? Did you find a pattern of a pattern? If you tried to use a formula, do you really understand the formula?

2. A protractor and a compass cost $3.00. If the protractor costs $.80 more than the compass, how much does each cost?

How many different ways did the class solve this problem? A fourth- or fifth-grade class might also come up with these same ways except for an algebra solution. Do you understand these different ways?

3. Looking in my backyard one day I saw some boys and dogs. I counted 24 heads and 72 feet. How many boys and how many dogs were in my backyard?

   How might children solve this problem with a picture?

4. There are four volumes of Charles Dickens's collected works on a shelf. The volumes are in order from left to right. The pages of each volume are exactly 2 inches thick. The covers of each volume are exactly 1/6 inch thick. A bookworm started eating at page 1 of Volume I and ate to the last page of Volume IV. What is the distance the bookworm traveled?

   A common method is to draw a picture and come up with the following solution: 4 sets of pages x 2 in. per set = 8 in. and 6 covers x 1/6 in. per cover = 1 in. so the total distance is 8 + 1 = 9 inches. This is a common solution given by other preservice teachers, but it is not how far the bookworm traveled. Some students interpret the cover to be a total of 1/6 in. for the entire book and they come up with 8 + 6 x 1/12 = 8 1/2 inches thick, which also is not how far the bookworm traveled. How far did the bookworm travel?

5. How many fence posts will it take to fence a rectangular field 250 feet by 300 feet if the fence posts are exactly 5 feet apart?

   Some students find the perimeter and divide by 5; other students find the perimeter, divide by 5, and then subtract 4 because they believe they have counted the corners twice; still other students make the rectangle into a line, divide the length of the line by 5, and then subtract 1. How can we check to see which method works?

6. If a snail is at the bottom of a well that is 100 feet deep and he climbs up 8 feet each day but slips back 5 feet each night, how long will it take him to climb out of the well?

   Some students may divide 100 by 3 as the first step to find the solution. How can we determine if this will help?

7. Which number does not belong?

   15 23 20 25

   Is there only one correct answer here?

8. How many rectangles are in this figure?

What is a rectangle? How can you help children make sure they get all the rectangles? Children do not find it natural to classify a square as a rectangle since they tend to classify objects into separate categories. Initially, for many children, a shape cannot be both a square and a rectangle—it is one or the other.

Problem Solving Steps

Almost every American elementary mathematics textbook uses Polya's four-step process, or a variation thereof, to help children become better problem solvers. His four steps are: Understand the Problem, Devise a Plan, Carry Out the Plan, and Look Back. Keep in mind that this is one way to approach problem solving. If it helps you, then it is a good way but there are other ways.

The next textbook example illustrates how children are taught Polya's four-step process.

From Math Central, Grade 2 (Houghton Mifflin: Boston 1998) p. 177.

The Open University (1988) in London's suggestions for problem solving are: STUCK, Good! RELAX and ENJOY it! Now something can be learned. Sort out What you KNOW and What you WANT, SPECIALIZE, GENERALISE, Make a CONJECTURE, Find someone to whom to explain why you are STUCK.

There is not one best approach to problem solving. Use the steps that help you and encourage children to use the steps that help them!

<u>Problem Solving Strategies - Heuristics</u>

Below is a list of some of the most common problem-solving strategies.

1 Guess and Check
2 Make a List or Table
3 Algebra
4 Work Backwards
5 Break into Smaller Parts
6 Draw a Picture
7 Act It Out
8 Look for a Pattern
9 Grind It Out – The Long Way
10 Take a Break and Try Again!

Oftentimes children will name one of the strategies by a method that a particular student used. For example, Grind It Out might be called Brian's Method. Children may also come up with strategies other than those on the list. Often students indicate that they used combinations of two and sometimes three strategies to solve a problem.

Homework: Find at least two problems in the supplement or in your textbook where you used each of these strategies.

These strategies have proven to be powerful tools in helping children and adults solve problems (Suydam, 1997). However, children's mathematics textbooks often tell students what strategies to use for each problem. (Your textbook may do this as well.)

Notice how in the following example children are told which strategy to use. Are there other strategies that children might have used to solve this problem?

From Silver Burdett Ginn, Grade 3 (New Jersey: Simon & Schuster, 1998) p. 60.

Such a “hint” may make the problem easy to solve—so easy, in fact, that it is not really a problem at all—but this ultimately limits students’ thinking. An important aspect of problem solving in many situations involves determining what strategy to try. Students cannot develop skill at assessing the applicability of various strategies if they are simply told what strategy to use. One way to help children become familiar with these strategies so that they can use them readily is to ask them to solve a set of problems in any way they choose and when they are finished have them go back and determine on which problems they used which strategy. This approach will help children personalize the strategies.

1.2 Patterns

Mathematics is about finding patterns. A formula is a generalization of a mathematical pattern that someone has created or discovered. The Pythagorean theorem, $a^2 + b^2 = c^2$, is a generalization of a pattern that was discovered about the relationship between the sides of right triangles. Children will not be asked to find patterns this complex but they will be finding simpler numerical and geometric patterns.

A major change in elementary textbooks is the increased emphasis on pattern finding. Recent reforms efforts have called for the introduction of algebraic reasoning in elementary school mathematics. (Algebraic reasoning will be discussed in more detail in a later section.) **Finding patterns can be classified as algebraic reasoning** because it is an activity that involves generalization and abstraction. New elementary mathematics textbooks will have even more pattern-finding activities because of the increased emphasis on algebraic reasoning.

Patterning is one of the first activities that kindergarten students do in school. A great source of pattern activities is Mathematics Their Way (MTW) by Mary Baratta-Lorton. MTW is a hands-on, activity-based approach to teaching children mathematics in K-2. An example of an MTW activity from an urban kindergarten class involved the teacher making an AB pattern with two colors of Unifix Cubes—blue, red, blue, red, blue, red, etc. She gave each child 10 cubes, 5 cubes of one color and 5 cubes of another color, and asked them to make her pattern with their cubes. Many students did, but one little boy had an AB<u>AABB</u>AB arrangement but said that it was an AB pattern. Patterning of this type may be obvious for adults, but it can be challenging for young children.

Another MTW example is for the children to snap and clap. The teacher may snap and clap twice and then ask the class to repeat the pattern. This is an ABB pattern. As children get better at finding patterns it is suggested that the teacher make patterns throughout the day and ask children to name or continue the pattern. For example, she could line the class for dismissal by boy, girl, boy, girl and then either ask the class what her pattern is or ask the class to finish lining up by this pattern. The brilliance of this type of pattern finding is that children are not looking at symbols on a page but are actively engaged in being and finding the patterns. Patterning activities can eventually be extended to three items, to growing patterns (e.g., ABABBABBBA), and to numbers 1,2,3, ….

The next textbook example is a patterning activity typical for first grade.

From Math Trailblazers, Grade 1 (Dubuque: Kendall/Hunt, 1997) p.62.

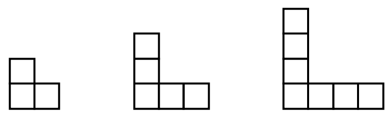
As children progress to higher grades their sophistication in finding patterns will also progress. They will be asked to find numerical and geometric patterns. An example of a numerical pattern-finding activity is the following:

__, 4, 7, __, 13, __, __

What is the Pattern? ___________

A common misconception of children in problems of this type is to make the first number in the pattern 3 because the pattern goes up by three.

Children will also be asked to find geometric patterns and generalize their thinking. For older students, problems that ask them to describe, extend, and make predictions about the numerical aspects of geometric patterns like the one below help develop the rudiments of algebraic thinking. Students may be asked to generalize a pattern and describe it symbolically. This introduces students to one of the primary uses of algebra—describing numerical relationships in a generalized form. Generalizing arithmetic is the basis for algebraic thinking.



Draw the next three figures in the sequence. How many tiles are needed for the 10th figure in the sequence?; the 20th figure?; the 100th figure?; the nth figure?

1.3 Mathematical Reasoning

Sense-Making

Your mathematics textbook may discuss inductive and deductive reasoning. These are powerful tools of mathematical thinking. However, children may not be able to characterize their mathematical thinking as following either line of reasoning (Reid, 2002). This is consistent with Piaget's assertion that young children are not capable of reflecting on their thinking at a level that would allow them to make such distinctions. Thus, in elementary school, our goal is not to teach inductive and deductive reasoning as if they were topics that must be covered.

Still, children should be encouraged to develop their mathematical reasoning (NCTM 2000). So what does this mean? Foremost, it means that mathematics should be a **sense-making activity**! That is, mathematics should not involve students blindly following rules that they do not understand. The mathematics children engage in should make sense to them. When it does they will begin to see mathematical relationships and how mathematics grows and is interconnected. Mathematical reasoning is promoted when children are asked to explain their mathematical thinking, to attempt to make sense of others' thinking, and to endeavor to resolve conflicting viewpoints that arise during discussions of mathematics.

As an illustration, can you explain why each of the numbers in problem #7 in Section 1.1 (15, 23, 20, 25) does not belong? Can you explain why verbally? In writing? On the other problems in this supplement or your textbook, did another student use a different method than yours? Did it make sense to you? Can you reconcile the two methods so that they both make sense? Often children will present their own way of solving problems—different from your way and the textbook. To teach mathematical reasoning you must try to make sense of students' explanations and, more importantly, you must try to help the class make sense of it as well. What questions might you ask the class to help them make sense of conflicting solution methods? Consider examples from this class either in your textbook or the supplement.

It is important for children to reason mathematically. There are numerous examples to illustrate that children are not reasoning mathematically and making sense of the mathematics. Some instructional strategies actually encourage children **not** to reason. Teaching children to look for key or cue words, a common practice when teaching word problems, is one such strategy. For example, children are taught that "altogether" means add and "left" means subtract. What do you think children who have been taught key words will do on the following problem: "Johnny walked 9 blocks. Then he turned left and walked 5 blocks. How many blocks did Johnny walk?" Children taught key words will often subtract (i.e., 9-5) and give 4 as their answer. The answer 4 makes no sense in this problem; it is not a reasonable answer given that he starts out by walking 9 blocks. For children who try to solve problems by looking for key words, **mathematics is not a sense-making activity!**

An example of a child who is making sense of mathematics is the first grader who solves the problem 3 + 4 = ? by reasoning, "I know that 3 + 3 = 6 and since 4 is 1 more than 3, the answer must be 1 more than 6, which is 7." Here the child is utilizing mathematical relationships that can be extended to other problems. For such a child, **mathematics is a sense-making activity**!

Incorporating Writing and Mathematics

Writing about mathematics is becoming very popular. Writing activities are one way to promote and explore children's mathematical reasoning. In addition, writing activities provide another means of assessment. More importantly, writing about mathematics also encourages children to reflect on their mathematical thinking and make their mathematical ideas more precise and communicable.

The next textbook example illustrates the emphasis to have children write about their mathematical thinking.

From Math Trailblazers, Grade 2 (Dubuque: Kendall/Hunt, 1997) p.251.

Chapter 1

Questions for Discussion and Review

1.1

1. Why study problem solving?
2. How did you feel about mathematics in school? Can you relate to the students quoted in the supplement?
3. What might you do so that problems really are problems for children and not routine tasks?
4. How do children solve problems differently than adults?
5. What does it mean to focus on the process when teaching mathematics? Give a specific example.
6. Describe one of your solution methods for a problem in this supplement or your textbook that you were particularly proud of.
7. What is the best way to solve a problem?
8. If a child is stuck, should the teacher immediately tell the child how to solve the problem?
9. How might you introduce problem-solving strategies to children?
10. Were Polya's four steps useful to you in solving problems? Would you teach them to children? Why or why not?

1.2

1. Describe a patterning activity that you might do in kindergarten or first grade.
2. Other than the quadratic formula, $a^2 + b^2 = c^2$, what are some other mathematical patterns?
3. Describe a pattern you found in solving problems from this supplement or the textbook.

1.3

1. Do you think elementary students, even fifth graders, are capable of understanding deductive reasoning?
2. What does the statement, "**mathematics should be a sense-making activity**," mean?
3. In your schooling, did mathematics always make sense? Describe a case where it did or did not.
4. Would you teach "key" words to children? Why or why not?

Chapter 2 Sets

2.1 Set Theory

Why study set theory? Set theory is a common topic in mathematics textbooks for elementary teachers. However, it is rarely addressed explicitly in elementary mathematics textbooks except in discussions of fractions of a set.

A Historical Perspective

In 1957 the Russians launched the first satellite to orbit the earth, Sputnik. This event created the "space race" and served as a catalyst for improving mathematics and science education in the United States. To address the crisis of confidence brought on by the launch of Sputnik, prominent mathematicians and scientists were consulted on how to improve K-12 education. They developed "New Math," which included set theory. Set theory was included because they believed that children could understand it and because it describes the fundamental structure of mathematics. From a mathematical standpoint, set theory provides the fundamental building blocks of our number system. The problem with presenting set theory in elementary school, as we later discovered through the works of Piaget and others, was that this is not how children learn mathematics. Today, "New Math" is no longer taught in elementary schools. But why then does almost every preservice elementary teacher study set theory?

What is Number?

In order to understand why we study set theory, consider the following problem: Define **number**, or, more specifically, define **the number 7**? Try to use the rules for defining words. For example, you cannot use the word in the definition. Hence, the word "number" or the word "seven" should not be used. In addition, the definition cannot contain circular reasoning. For example, you should not define 7 as 1 more than 6 because now 6 and 1 must also be defined.

Number is difficult to define and in mathematics set theory is used to define it. More importantly, number is a difficult concept that we expect 4- and 5-year-olds to master. If we cannot define number, or we use higher-level mathematics to define number, how can we expect 4- and 5-year-olds to understand number let alone start adding and subtracting numbers?

The work of Piaget (1945/1962) and others describes how children come to know number (von Glaserfeld, 1995; Sinclair, Siegrest & Sinclair, 1983; Kamii, 2000).
Research shows that numbers are not properties of objects, but of sets that we construct in our minds. Children's knowledge of "four," for instance, grows out of activity such as contrasting the counting of sets having four objects with the counting of sets having three and five objects. That is, children begin to construct an understanding about numbers as they develop the ability to make and count collections of objects. As their counting processes develop, they abstract mathematical concepts from their activity. The

intertwining of their counting ability and their intuitive notions about sets (collections of objects) forms the foundation not only for children's construction of a concept of number, but also for their development of an understanding of the basic operations with numbers. For example, "2 + 3" involves combining two sets, "7 – 4" involves removing objects from a set or comparing two sets, "2 x 4" can be viewed as two sets of four, and "12 ÷ 3" can be thought of as asking how many sets of three there are in 12 or how many there would be in each set if 12 were divided into three sets.

Reasons for Studying Set Theory

The preceding discussion suggests the following reasons why it is important for prospective teachers to study set theory:

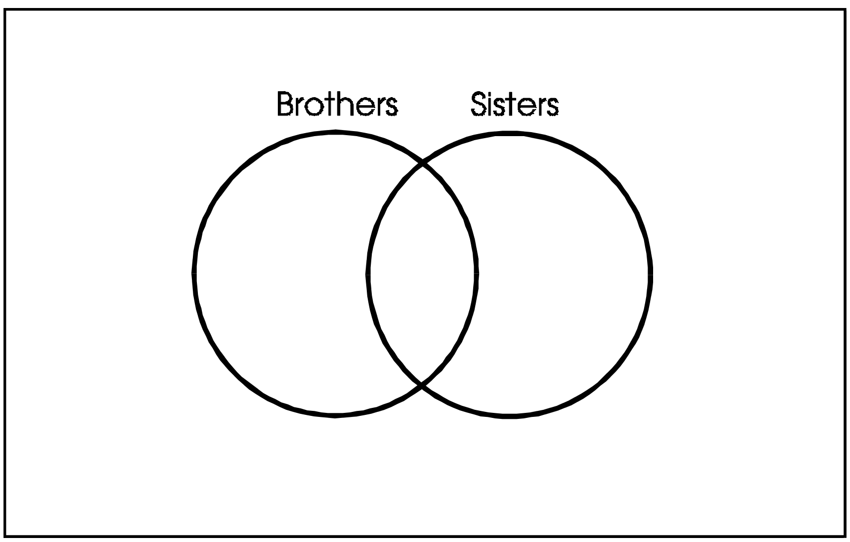
- Set theory enables mathematicians to define number and the four basic operations on number.
- Studying set theory may provide insights into the structure of our number system and higher-level mathematics.
- As future teachers it is important to understand the basis for the mathematics that will be taught to children.
- Much of children's mathematics is based on their intuitive notions about sets of objects.

The next textbook example describes how to assess if children have developed a 1-1 correspondence between number and the elements of a set.

From Mathematics Their Way (Menlo Park: Addison-Wesley, 1976) p. 92.

2.2 Venn Diagrams

Venn diagrams are sometimes used in the elementary school to help children organize and reflect on their thinking. An example of the use of a Venn diagram that occurred in a second-grade classroom involved children's siblings. Two intersecting circles in a rectangle were drawn on a poster board. The circles were labeled "Brothers" and "Sisters," respectively. The students put their names in the appropriate section according to whether they had brothers only, sisters only, brothers and sisters, or had no brothers and sisters. The intersection of the two circles was for those who had both brothers and sisters, the rest of the Brothers circle was for those who had brothers only, the rest of the Sisters circle was for those who had sisters only, and the space outside both circles was for children with no brothers or sisters. This poster remained on the bulletin board for several months. A valuable learning experience occurred when one student had a new baby in the family and moved his name to another region of the diagram. This was a nice activity because it related directly to the children. Further, because the teacher left the Venn diagram on the bulletin board for several months, children had the opportunity to reflect on the Venn diagram and the use of sets throughout the school year. This was not just a one-day activity but also an activity that offered continuous learning opportunities!



At a more advanced level, Venn diagrams can also help students organize their thinking and solve problems involving three intersecting sets.

Chapter 2

Questions for Discussion and Review

2.1

1. Why study set theory?

2. Why is number so difficult to define?

3. How do you think children first come to understand number?

4. How can set theory be used to describe a) addition; b) subtraction; c) multiplication?

2.2

1. How can using Venn diagrams help children organize their thinking?